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„Smooth Functorial Field Theory and the Geometric Cobordism Hypothesis“

verfasst von / submitted by

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Notation and Conventions

Throughout, the calligraphic letters $\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{D},\mathscr{E}$ and so on will usually refer to some flavour of category (1-category, bicategory, ∞-category etc.). Fracture symbols $\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{F},\mathfrak{U},\mathfrak{N}$ and so on will usually refer to some flavour of functors. Let $\mathscr{C}$ be a category. The associated Hom-set for two given objects c, c' is denoted by $\mathscr{C}(c,c')$ and the corresponding Hom-set functor is written as $\mathscr{C}(-,-)\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{C}\to \mathrm{Set}$. The covariant Yoneda embedding

$$よ\colon \mathscr{C}\to \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}, \qquad c\mapsto \mathscr{C}(-,c)$$

is denoted by the japanese letter よ which is phonetically given by "yo". The contravariant Yoneda embedding

$$ふ\colon \mathscr{C}^{\mathrm{op}}\to \mathrm{Set}^{\mathscr{C}}, \qquad c\mapsto \mathscr{C}(c,-)$$

is denoted by the japanese letter ふ which is phonetically given by "fu". To make explicit to which category such a Yoneda embedding corresponds to we shall sometimes write $よ_{\mathscr{C}}$ and $ふ_{\mathscr{C}}$. For functors $\mathfrak{F},\mathfrak{U}\colon \mathscr{C}\to\mathscr{D}$ we shall sometimes write

$$\mathscr{D}(\mathfrak{F},-)\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{D}\to \mathrm{Set}, \qquad \mathscr{D}(-,\mathfrak{U})\colon \mathscr{C}^{\mathrm{op}}\to \mathrm{Set}, \qquad \mathscr{D}(\mathfrak{F},\mathfrak{U})\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{C}\to \mathrm{Set}$$

for the induced Hom-functors given by the compositions

$$\mathscr{C}^{\mathrm{op}}\times\mathscr{D} \xrightarrow{\mathfrak{F}^{\mathrm{op}}\times \mathrm{id}} \mathscr{D}^{\mathrm{op}}\times\mathscr{D} \xrightarrow{\mathscr{D}(-,-)} \mathrm{Set}$$

$$\mathscr{D}^{\mathrm{op}}\times\mathscr{C} \xrightarrow{\mathrm{id}\times\mathfrak{U}} \mathscr{D}^{\mathrm{op}}\times\mathscr{D} \xrightarrow{\mathscr{D}(-,-)} \mathrm{Set}$$

$$\mathscr{C}^{\mathrm{op}}\times\mathscr{C} \xrightarrow{\mathfrak{F}^{\mathrm{op}}\times\mathfrak{U}} \mathscr{D}^{\mathrm{op}}\times\mathscr{D} \xrightarrow{\mathscr{D}(-,-)} \mathrm{Set}$$

where $\mathfrak{F}^{\mathrm{op}}\colon \mathscr{C}^{\mathrm{op}}\to\mathscr{D}^{\mathrm{op}}$ denotes the opposite functor of $\mathfrak{F}$.

Abstract

> The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it... yet it finally surrounds the resistant substance.
>
> Alexander Grothendieck, Récoltes et semailles, 1985–1987, pp. 552-3-1 The Rising Sea

The best gauge to determine how good a physical theory really is, is by looking at the predictions the theory provides and comparing the resulting numbers with real-world experiments. With regards to this measure, Quantum Field Theory (QFT) is probably the best physical theory there is to this date. Yet, a fully general mathematically rigorous formulation of (non-topological) QFT is missing. In this thesis we will study the functorial approach to QFT. More specifically, we will study the specific approach taken in [16] and [17]. In particular, [17] provides a classification theorem for *smooth spaces* of QFTs, referred to as the *geometric cobordism hypothesis*. The geometric cobordism hypothesis is a generalization of the *topological cobordism hypothesis*, which can be traced back to Baez and Dolan (1995), and was later rigorously formulated by Lurie in [24].

This work aims to accomplish two goals. The first of these is to provide a self-contained introduction to the somewhat intimidating realm of smooth functorial field theory. This is why the first six chapters of this thesis are devoted to the study of notions like simplicial homotopy theory, enriched category theory, model categories, ∞-sheaves, ∞-categories etc. The only prerequisites to be had in order to be able to follow the material is a good understanding of (ordinary) category theory (algebraic topology is helpful, but not needed). The second goal of this thesis is to provide a better understanding of the construction of smooth bordism ∞-categories endowed with geometric structures as defined in [16]. This is done by first providing the rigorous construction of these objects, and by then looking at some low-dimensional examples thereof. With that in hand, a smooth field theory with some prefixed geometric structure should then just be an ∞-functor from the given smooth bordism ∞-category to some ∞-category of values. From there, we will consider smooth spaces of field theories with prefixed geometry and explain the geometric cobordism hypothesis, which, roughly put, states that such a space of field theories is equivalent to "morphisms" from the given geometric structure to the *maximal ∞-subgroupoid of fully dualisable objects* of the target ∞-category.

Abstrakt (Deutsch)

Wie gut eine physikalische Theorie wirklich ist, lässt sich am besten feststellen, wenn man die Vorhersagen der Theorie betrachtet und die daraus resultierenden Zahlen mit realen Experimenten vergleicht. Im Hinblick auf diesen Maßstab ist die Quantenfeldtheorie (QFT) wahrscheinlich die beste physikalische Theorie, die es bis heute gibt. Dennoch fehlt eine vollständig allgemeine, mathematisch rigorose Formulierung der (nicht-topologischen) QFT. In dieser Arbeit werden wir den funktoriellen Ansatz zur QFT studieren. Genauer gesagt, werden wir den spezifischen Ansatz in [16] und [17] untersuchen. Insbesondere liefert [17] ein Klassifikationstheorem für *glatte Räume* von QFT, das als die *geometrische Kobordismushypothese* bezeichnet wird. Die geometrische Kobordismushypothese ist eine Verallgemeinerung der *topologischen Kobordismushypothese*, welche auf Baez und Dolan (1995) zurückzuführen ist, und später von Lurie in [24] rigoros formuliert wurde.

Mit dieser Arbeit sollen zwei Ziele erreicht werden. Das erste Ziel besteht darin, eine in sich geschlossene Einführung in das etwas einschüchternde Gebiet der glatten funktoriellen Feldtheorie zu geben. Aus diesem Grund sind die ersten sechs Kapitel dieser Arbeit dem Studium von Gebieten wie simplizialer Homotopietheorie, angereicherter Kategorientheorie, Modellkategorien, ∞-Garben, ∞-Kategorien usw. gewidmet. Die einzigen Voraussetzungen, die man haben muss, um dem Material folgen zu können, ist ein gutes Verständnis der (gewöhnlichen) Kategorientheorie (algebraische Topologie ist hilfreich, aber nicht erforderlich). Das zweite Ziel dieser Arbeit ist es, ein besseres Verständnis für die Konstruktion von glatten Bordismus-∞-Kategorien zu schaffen, die mit geometrischen Strukturen ausgestattet sind, wie sie in [16] definiert sind. Dies geschieht, indem wir zunächst die rigorose Konstruktion dieser Objekte bereitstellen und dann einige niedrig-dimensionale Beispiele dafür betrachten. Eine glatte Feldtheorie mit einer vordefinierten geometrischen Struktur sollte dann einfach ein ∞-Funktor von der gegebenen glatten Bordismus ∞-Kategorie zu einer ∞-Kategorie von Werten sein. Von dort aus werden wir glatte Räume von Feldtheorien mit gegebener Geometrie betrachten und die geometrische Kobordismus-Hypothese erklären, die, grob gesagt, besagt, dass ein solcher Raum von Feldtheorien äquivalent zu Morphismenvon der gegebenen geometrischen Struktur zum *maximalen ∞-Untergruppoid von vollständig dualisierbaren Objekten* der Ziel-∞-Kategorie ist.

Contents

Acknowledgements

I would like to offer a lighthearted apology to Professor Nils Carqueville for the rather hefty length of my thesis, which seemed to have taken on a life of its own.

In particular, I would like to express my heartfelt gratitude to Nils for his exceptional guidance and support as my thesis supervisor. His patience and the freedom he provided me with were instrumental in the shaping of my thesis. I am also immensely grateful to Professor Dmitri Pavlov who not only responded to all those lengthy emails I sent his way, but also devoted most of his time to discussions with me when he was visiting our research group in Vienna. Moreover, special thanks go to Konrad Martinek with whom I had a lot of insightful discussions on ∞-categories.

Finally, I am grateful to my family and friends for their unwavering encouragement.

Thank you all for your significant contributions to my academic journey.

1. Introduction

The integration of mathematical concepts from homotopy theory into the study of physics is increasingly recognized as a valuable approach for tackling fundamental problems. This framework offers powerful tools that facilitate a deeper understanding of complex phenomena and may pave the way for solutions to currently intractable physical questions. One such compelling reason to go *homotopy coherent* lies in the quest for a rigorous formulation of quantum field theory. The lack of a proper definition for the Feynman path integral, a fundamental concept in this field, hinders its mathematical foundation. Homotopy theory (and in turn higher category theory) offers a compelling approach to tackle this issue. Its tools and techniques enable us to construct a solid framework for the Feynman path integral, paving the way for a rigorous treatment of quantum field theory. In this thesis we will first provide a self-contained introduction to some of the mathematical machinery that is common practice within the subject of *smooth functorial field theory*. After that, we will concern ourselves with the study of smooth ∞-bordism categories along with the classification theorem regarding smooth field theories called the *geometric cobordism hypothesis* (as stated and proved in [17]). The geometric cobordism hypothesis is a generalization of the topological cobordism hypothesis which dates back to Baez and Dolan (1995) and was popularized by Jacob Lurie in his paper [24].

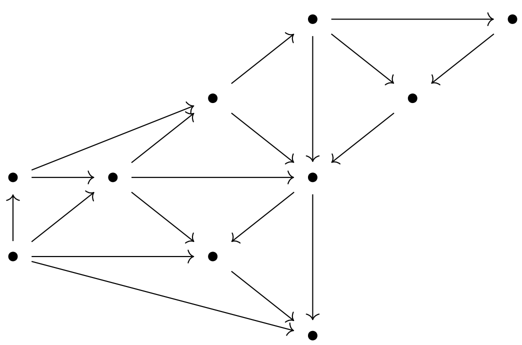
In this introduction we shall quickly sketch each individual part of the thesis and give some of the main ideas:

1.1. On Part I.

As already mentioned, the first part of the thesis is on prerequisites.

1.1.1. *Simplicial Sets.*

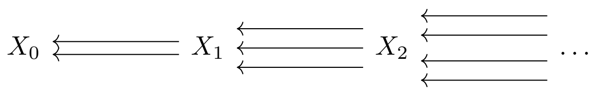
In the first chapter we introduce the notion of a simplicial set. Good references on this subject are [15, 33, 21]. The idea of a simplicial set is that, up to homotopy, it is just as good a notion for space as a topological space, yet simplicial sets are far easier to work with as their nature is combinatorial and every such simplicial set may be obtained by glueing triangles, tetrahedra and higher dimensional versions of these. For example, a simplicial set could look like:



More precisely, a simplicial set $X_\bullet$ is a list of sets

$$X_0, X_1, X_2, X_3, \ldots$$

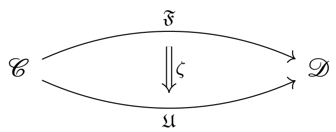
indexed by the natural numbers. Moreover, there are so called *face* and *degeneracy* maps $d_i^{(n)}: X_n \to X_{n-1}$ and $s_i^{(n)}: X_n \to X_{n+1}$. These maps are then subject to some relations. One can then diagrammatically depict a simplicial set by



where we only drew the face maps. The question now is why would we be interested in having a combinatorial variant to topological spaces. First of all, it turns out that the category of simplicial sets is rich enough so that the theory of category theory, i.e. the category of small categories, is fully faithfully contained within the category of simplicial sets. This is seen by taking the *nerve* of a category. Indeed, for a category $\mathscr{C}$, denote by $\mathfrak{N}\mathscr{C}_\bullet$ the collection of sets

$$\left\{\mathfrak{N}\mathscr{C}_n\right\}_{n\in\mathbb{N}} := \left\{n\text{-tuples of composable morphisms in } \mathscr{C}\right\}$$

where 0-tuples of composable morphisms are identified with objects in the category $\mathscr{C}$. It is shown that this nerve construction extends to yield a functor between the category of small categories and the category of simplicial sets $\mathfrak{N}\colon \mathrm{Cat} \to \mathrm{sSet}$ which is fully faithful. Moreover, the category of simplicial sets allows for the notion of simplicial homotopy theory, which is equivalent in the proper sense to the homotopy theory of topological spaces. In that framework, the following picture of two functors and a natural transformation between these



is equivalently given by saying that we have a simplicial homotopy between the respective nerves of functors. In particular, after having developed more machinery it will be noted that the category of simplicial sets and its associated homotopy theory is a model for the notion of ∞-groupoids.

1.1.2. *Model Categories.* After having concerned ourselves with simplicial sets we come to the concept of a model category. A good reference for this field is [19]. At that point we realize that the category of simplicial sets, just like the category of topological spaces is a *model category.* Roughly put, model category theory is the study of abstract homotopy theories. The objects of study in this field are the so-called model categories, which can be thought of as categories which allow for a proper notion of deformation, that is, some object A may be deformed (is homotopic) to another object B. More precisely, a model category is a category $\mathscr{C}$ with distinguished classes of morphisms $\mathscr{W}_\mathscr{C}, \mathrm{Fib}_\mathscr{C}$ and $\mathrm{Cof}_\mathscr{C}$, referred to as *weak equivalences, fibrations* and *cofibrations* which are subject to some axioms. Here the canonical example to think of is the model category of topological spaces, which, for example, has as its set of weak equivalences the set of weak homotopy equivalences. There now has to be some notion of equivalence for model categories. This concept is referred to as a *Quillen equivalence* of model categories. The existence of such a Quillen equivalence between two model categories says that the respective homotopy theories are precisely the same. The most prominent such equivalence is the adjunction between the *geometric realization functor* $|-|$ and the *fundamental ∞-groupoid functor* $\Pi_{\leqslant\infty}$:

$$\mathrm{sSet} \underset{\Pi_{\leqslant\infty}}{\overset{|-|}{\underset{\perp \mathrm{Quillen}}{\rightleftarrows}}} \mathrm{Top}$$

Roughly put, the left adjoint $|-|$ takes a simplicial set and realizes it as a topological space, e.g., the geometric realization of the above depiction of a simplicial set is given by filling out the triangles to proper triangles with area in $\mathbb{R}^2$ and then glueing them along their edges. The right adjoint $\Pi_{\leqslant\infty}$ takes a topological space X and maps it to the simplicial set $\Pi_{\leqslant\infty}X_\bullet$ for which we have:

$$\Pi_{\leqslant\infty}X_0 := \text{points of } X$$

$$\begin{aligned}\Pi_{\leqslant\infty}X_1 &\coloneqq \text{paths in } X\\ \Pi_{\leqslant\infty}X_2 &\coloneqq \text{homotopies of paths in } X\\ &\vdots\\ \Pi_{\leqslant\infty}X_n &\coloneqq \text{homotopies of homotopies ... of paths in } X\end{aligned}$$

Finally, we will talk about the notion of what it means for a functor between model categories to be *homotopical* (i.e. it preserves all the homotopical information of our given model category), and we will discuss that if our functor fails to be homotopical, we may still have a chance by taking the respective left or right derived functors (if they exist), which are the closest homotopical approximations to our initial functor. In particular, this will give rise to the notion of homotopy limit and colimit functors, which are right and left derived functors of the usual limit and colimit functors, respectively.

1.1.3. ∞-*Categories.* Good references on ∞-categories are given by [26] and the respective parts in [8] and [24]. An ∞-category should be the precise mathematical entity of a higher dimensional category in the sense that it should not only have objects and morphisms, but also morphisms between morphisms and morphisms between morphisms between morphisms and so on. In other words, an ∞-category $\mathscr{C}$ should be a collection of sets

$$\mathscr{C}_0, \mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \dots$$

where $\mathscr{C}_0$ is the set of objects while $\mathscr{C}_n$ denotes the set of n-morphisms. In particular, there should be source and target maps for each individual level of morphisms

$$s^{(n)}\colon \mathscr{C}_n \to \mathscr{C}_{n-1}, \qquad t^{(n)}\colon \mathscr{C}_n \to \mathscr{C}_{n-1}$$

as well as composition maps

$$c^{(n)}\colon \mathscr{C}_n \times_{\mathscr{C}_{n-1}} \mathscr{C}_n \to \mathscr{C}_n$$

where $\mathscr{C}_n \times_{\mathscr{C}_{n-1}} \mathscr{C}_n$ denotes composable n-morphisms with regards to the aforementioned source and target maps. Moreover, we need units with respect to composition in each layer:

$$u^{(n)}\colon \mathscr{C}_{n-1} \to \mathscr{C}_n$$

A canonical example of such an entity is the *fundamental* ∞-*groupoid* $\Pi_{\leqslant\infty}X$ for some topological space X. We recall that objects are given by points, 1-morphisms are given by paths, 2-morphisms are given by homotopies of paths and so on. Composition of morphisms is given by concatenation of paths, homotopies, etc. The unit maps are the constant paths, homotopies, and so on, while source and target maps are the obvious choices. Upon further inspection we realize that composition of paths, homotopies, ... is not unique, but only unique up to homotopy. This is a general theme when it comes to ∞-categories: Composition will not be assumed to be unique, but only unique up to homotopy. The reason for this is that *strict* ∞-categories do not capture the most interesting examples that might pop up in practice (as for example topological spaces). In particular, when we consider $\Pi_{\leqslant\infty}X$ yet again we notice that all morphisms (in every layer) have an inverse up to (higher) homotopy. This is the reason for calling $\Pi_{\leqslant\infty}$ the fundamental ∞-groupoid functor, as, by definition, $\Pi_{\leqslant\infty}X$ is an ∞-*groupoid* (all morphisms have inverses up to homotopy) for all topological spaces X. In fact, *Grothendieck's homotopy hypothesis* states that any sensible notion of ∞-groupoids should imply that ∞-groupoids are precisely the *homotopy types* of topological spaces. Using this as the literal definition is then saying that any ∞-groupoid is realized by considering $\Pi_{\leqslant\infty}X$ for some suitable topological space X.

An ∞-groupoid is also often referred to as $(\infty,0)$-category, as although the category at hand has infinitely many layers, 0 layers of them have non-invertible morphisms. An *(∞,d)-category* on the other hand also has infinitely many layers, but only the first d layers of morphisms may contain non-invertible morphisms, while all $(d+1),(d+2),\ldots$-morphisms are invertible up to homotopy. We can use an inductive formulation to roughly define the notion of (∞,d)-category. An *$(\infty,1)$-category* $\mathscr{C}$ is the data of a set of objects $\mathscr{C}_0$, and an *∞-groupoid (a space) of 1-morphisms* $\mathscr{C}_1$. The points $\mathscr{C}_{1,0} := \Pi_{\leqslant\infty}(\mathscr{C}_1)_0$ in this space of morphisms are the 1-morphisms in $\mathscr{C}$, the paths are the 2-morphisms and so on. An $(\infty,2)$-category $\mathscr{C}$ is then the data of a set of objects $\mathscr{C}_0$ as well as an *$(\infty,1)$-category of 1-morphisms* $\mathscr{C}_1$. The 1-morphisms in $\mathscr{C}$ are the objects of $\mathscr{C}_1$, while the higher morphisms are given by the fundamental ∞-groupoid of the space of 1-morphisms of $\mathscr{C}_1$, denoted by $\Pi_{\leqslant\infty}(\mathscr{C}_{1,1})$. Continuing in this way, an *(∞,d)-category* is the data of a set of objects $\mathscr{C}_0$ and an *$(\infty,d-1)$-category $\mathscr{C}_1$ of 1-morphisms*. We then remind ourselves that the model category of simplicial sets and the model category of topological spaces have the same homotopy theories (this is witnessed by the aforementioned Quillen equivalence). In that sense, instead of using topological spaces as a definition for ∞-groupoid we can also just use very nice simplicial sets called *Kan complexes* as our model for ∞-groupoids. Using again an inductive procedure as before, we quite naturally arrive at the definition of *d-fold complete Segal spaces* which present a fully rigorous simplicial version of (∞,d)-categories.

This is not the end of the road however, we want to define (∞,d)-categories with extra structure. In fact, we will make sense of the notion of *symmetric monoidal (∞,d)-category*, which roughly put is an (∞,d)-category $\mathscr{C}$ equipped with a *tensor ∞-functor* $\otimes\colon \mathscr{C}\times\mathscr{C}\to\mathscr{C}$, that is, a collection of maps

$$\Big\{\otimes_n\colon \mathscr{C}_n\times\mathscr{C}_n\to\mathscr{C}_n\Big\}_{n\in\mathbb{N}}$$

where $\mathscr{C}_n$ denotes the set of n-morphisms in $\mathscr{C}$ (if $n=0$, $\mathscr{C}_n$ is the set of objects), which satisfy coherence conditions. Morally, $\otimes$ tells us how to multiply objects, 1-morphisms, ... in $\mathscr{C}$. We still do not stop there and define the notion of *smooth symmetric monoidal ∞-categories*. In order to give the idea, let Cart be the category of *cartesian spaces*, which has as its set of objects open subsets U of $\mathbb{R}^n$, for some n, such that U is smoothly diffeomorphic to $\mathbb{R}^n$. Morphisms in Cart are then simply smooth maps between cartesian spaces. Vaguely put, a smooth symmetric monoidal (∞,d)-category $\mathscr{C}$ is then nothing else than a contravariant functor into symmetric monoidal (∞,d)-categories

$$\mathscr{C}\colon \mathrm{Cart}^{\mathrm{op}}\to\mathrm{Cat}^{\otimes}_{(\infty,d)},\qquad U\mapsto\mathscr{C}(U)$$

such that for any *good open cover* $\{V_i\}_{i\in I}$ of U in Cart we have that $\mathscr{C}(U)$ may be given as the homotopy limit of the diagram

$$\prod_{i\in I}\mathscr{C}(V_i) \rightrightarrows \prod_{i_0,i_1\in I}\mathscr{C}(V_{i_0}\cap V_{i_1}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \prod_{i_0,i_1,i_2\in I}\mathscr{C}(V_{i_0}\cap V_{i_1}\cap V_{i_2}) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots$$

Morally, this says that $\mathscr{C}$ is an *∞-sheaf of symmetric monoidal (∞,d)-categories*, that is, local higher dimensional information can be glued to obtain higher dimensional global information. Finally, we will define the notion of *full dualizability* for smooth symmetric monoidal (∞,d)-categories, which more or less says that each layer (also all objects) have *adjoints* (*duals*). One may then collect all this information to arrive at the model category of smooth symmetric monoidal categories with duals $\mathscr{C}^\infty\mathrm{Cat}^{\otimes,\dagger}_{(\infty,d)}$.

1.2. **On Part II.** This part of the thesis is based upon the works [16] and [17], as well as on private communication with Dmitri Pavlov.

If we interpret an ∞-category as a language in the literal sense, then a quantum field theory would be a translation from some language of spacetime to some language of values. More precisely, a *d-dimensional smooth quantum field theory* is a smooth symmetric monoidal ∞-functor from some smooth (∞, d)-category of bordisms to some smooth (∞, d)-category of values. The first goal of this part of the thesis is to properly introduce these smooth bordism categories.

1.2.1. *Smooth Bordism ∞-categories.* Since we want our bordism categories to be endowed with some geometry (e.g. Riemannian metrics), we start off this chapter by properly introducing the notion of a *d-dimensional geometric structure with isotopies* as discussed in [16]. Very roughly put, a geometric structure is an ∞-sheaf, on the *simplicially enriched* category $\mathfrak{FEmb}_d$ of fiberwise embeddings, valued in ∞-groupoids. To give at least some explanation here, the category $\mathfrak{FEmb}_d$ has as its set of objects fiberwise d-dimensional submersions $p\colon M \twoheadrightarrow U$, where M is a smooth manifold while $U \in \mathrm{Cart}$ is a cartesian space. After discussing some examples in this formalism, we will move on to the definition of the smooth (∞, d)-category of bordisms with geometry $\mathbf{S}$, denoted $\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)}$, where $\mathbf{S}$ is some d-dimensional geometric structure. The construction is roughly as follows:

- Objects of $\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)}$ are smooth families of disjoint unions of points in a d-dimensional manifold equipped with a d-dimensional germ of the given geometric structure $\mathbf{S}$.
- 1-morphisms are smooth families of 1-dimensional manifolds with boundaries between smooth families of disjoint unions of points embedded within a d-dimensional manifold, which is again equipped with a d-dimensional germ of the given geometric structure.
- ...
- d-morphisms are smooth families of d-dimensional manifolds with corners which are equpped with a d-dimensional germ of the given geometric structure $\mathbf{S}$.
- $d + 1$-morphisms are smooth families of isotopies of diffeomorphisms between d-dimensional manifolds with corners.
- $d + 2$-morphisms are smooth families of isotopies of isotopies ... etc.
- ...

After giving a precise definition of the above, we will carry on to investigate properties of the assignment $\mathbf{S} \mapsto \mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)}$. It will turn out that this gives rise to a functor

$$\mathfrak{Bord}^{(-)}_{(\infty,d)}\colon \mathfrak{Struct}_d \to \mathscr{C}^\infty\mathrm{Cat}^{\otimes,\dagger}_{(\infty,d)}$$

from the category of geometric structures to the category of smooth symmetric monoidal (∞, d)-categories with duals, which itself will be an ∞-*cosheaf*. This is the so-called *locality property*, which may also be phrased by saying that $\mathfrak{Bord}^{(-)}_{(\infty,d)}$ preserves homotopy colimits. Finally, we will discuss the symmetric monoidal structure of $\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)}$ and its duals as well as consider specific examples including Riemannian bordism categories.

1.2.2. *Smooth Functorial Field Theories.* Finally, we can say more clearly what a smooth functorial field theory with geometry $\mathbf{S}$ really should be, namely a smooth symmetric monoidal ∞-functor $\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)} \to \mathscr{C}$, where $\mathscr{C}$ is some smooth symmetric monoidal (∞, d)-category of values. We may then first state the content of the *framed geometric cobordism hypothesis*. To this end, we first realize that if we take the (enriched) Yoneda embedding $よ(\mathbb{R}^d \times U \twoheadrightarrow U)$ of the canonical projection map $(\mathbb{R}^d \times U \twoheadrightarrow U) \in \mathfrak{FEmb}_d$, then the resulting geometric structure models smooth U-families of framings for the given input manifolds. We can then consider the smooth symmmetric monoidal (∞, d)-functor category

$$\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,d)}^{よ(\mathbb{R}^d \times U \twoheadrightarrow U)}, \mathscr{C})$$

for $\mathscr{C}$ some smooth symmetric monoidal (∞, d)-category with duals. The first statement of the framed geometric cobordism hypothesis says that the assignment $U \mapsto \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,d)}^{よ(\mathbb{R}^d \times U \twoheadrightarrow U)}, \mathscr{C})$ is actually a functor valued in smooth symmetric monoidal ∞-groupoids:

$$\mathrm{Cart}^{\mathrm{op}} \to \mathscr{C}^{\infty}\mathrm{Grpd}_{\infty}^{\otimes}, \qquad U \mapsto \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,d)}^{よ(\mathbb{R}^d \times U \twoheadrightarrow U)}, \mathscr{C})$$

The second and even more important statement is that evaluation at a U-family of points gives rise to an equivalence of ∞-categories

$$\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,d)}^{よ(\mathbb{R}^d \times U \twoheadrightarrow U)}, \mathscr{C}) \xrightarrow{\simeq} \mathfrak{Map}(U, \mathscr{C}^{\times})$$

where $\mathscr{C}^{\times}$ is the maximal full ∞-subgroupoid in $\mathscr{C}$, while $\mathfrak{Map}(U, \mathscr{C}^{\times})$ denotes a smooth symmetric monoidal ∞-groupoid of maps from U to $\mathscr{C}^{\times}$. If one forgets the smoothness property on both sides of the equivalence (by evaluating at the singleton cartesian space $\mathbb{R}^0$) one arrives at

$$\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,d)}^{よ(\mathbb{R}^d \times U \twoheadrightarrow U)}, \mathscr{C})(\mathbb{R}^0) \simeq \mathscr{C}^{\times}(U)$$

The general *geometric cobordism hypothesis* makes similar claims. First of all, we have a functor

$$\mathfrak{Struct}_d^{\mathrm{op}} \to \mathscr{C}^{\infty}\mathrm{Grpd}_{\infty}^{\otimes}, \qquad \mathbf{S} \mapsto \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,d)}^{\mathbf{S}}, \mathscr{C})$$

which has values in smooth symmetric monoidal ∞-groupoids. Moreover, we have an equivalence of ∞-categories

$$\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,d)}^{\mathbf{S}}, \mathscr{C}) \xrightarrow{\simeq} \mathfrak{Map}_{\mathfrak{FEmb}_d}(\mathbf{S}, \mathscr{C}_d^{\times})$$

where, morally speaking, $\mathscr{C}_d^{\times}$ is identified with $\mathscr{C}^{\times}$ (this is not quite true), while $\mathfrak{Map}_{\mathfrak{FEmb}_d}(\mathbf{S}, \mathscr{C}_d^{\times})$ denotes a smooth symmetric monoidal ∞-groupoid of maps from the geometric structure $\mathbf{S}$ to $\mathscr{C}_d^{\times}$.

2. Simplicial Homotopy Theory

Education never ends, Watson. It is a series of lessons, with the greatest for the last.

Sherlock Holmes (Sir Arthur Conan Doyle)

This chapter is based on [11], [28] and [15].

Simplicial sets are a powerful tool in algebraic topology that provide a combinatorial framework for studying spaces. They are a way to encode the topology of a space using a collection of abstract building blocks called simplices, which are higher-dimensional generalizations of triangles and tetrahedra. In this chapter, we will explore the basic concepts and properties of simplicial sets, including their given homotopy theory. In between, we shall also introduce the notion of ends and coends as they will provide a powerful tool throughout.

2.1. **A Theory of Simplices.**

Definition 2.1. The *simplex category* Δ has

- objects $[n] = \{0, 1, \dots, n\}$ for $n \in \mathbb{N}$, and
- morphisms $f\colon [n] \to [m]$ are order preserving maps, i.e., $f(i) \leqslant f(j)$ for all $i \leqslant j$.

The simplex category is a combinatorial framework for the collection of topological spaces that is made up of the standard topological n-simplices. More precisely, there is a functor

$$\Delta \xrightarrow{\;|-|\;} \mathrm{Top}, \qquad [n] \longmapsto |\Delta^n|$$

where generating

$$|\Delta^n| := \left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geqslant 0, \sum x_i = 1\right\}$$

is endowed with the subspace topology induced from the Euclidean topology on $\mathbb{R}^n$. On the other hand, a morphism $f\colon [n] \to [m]$ in the simplex category is mapped to the continuous map $|f|\colon |\Delta^n| \to |\Delta^m|$ given by

$$|\Delta^n| \ni x \mapsto \Big(\sum_{s\in[n]\colon f(s)=i} x_s\Big)_{i=0}^{m} \in |\Delta^m| \tag{1}$$

In order to give a framework for more general topological spaces, one introduces the notion of a simplicial set, or more generally the notion of a simplicial object. This is motivated upon noticing that many topological spaces can be obtained by glueing n-simplices.

Definition 2.2. Let $\mathscr{C}$ be a category.

- A *simplicial object* in $\mathscr{C}$ is an object in the functor category $\mathscr{C}^{\Delta^{\mathrm{op}}}$.
- A *simplicial set* is a simplicial object in Set. The category of simplicial sets will be denoted by $\mathrm{sSet} := \mathrm{Set}^{\Delta^{\mathrm{op}}}$.
- For $X \in \mathrm{sSet}$ we write $X_n := X([n])$ and $X_f := X(f)$ for any object $[n]$ and any morphism f in the simplex category Δ.
- An element $x \in X_n$ is called an *n-simplex* of the simplicial set X.

It is immediate from the Yoneda lemma that we have an embedding $\Delta \hookrightarrow \mathrm{sSet}$ given by $[n] \mapsto \Delta^n := \Delta(-, [n])$. In particular, for $X \in \mathrm{sSet}$ we have

$$X_n \cong \mathrm{sSet}(\Delta^n, X)$$

This means that any n-simplex $x \in X_n$ uniquely corresponds to a simplicial map (natural transformation) $x\colon \Delta^n \to X$.

The category Δ has a generating set of morphisms. Indeed, we may define

$$\textit{coface maps} \qquad \forall n \geqslant 0\colon [n-1] \overset{d^i}{\rightarrowtail} [n] \qquad d^i(k) = \begin{cases} k, & \text{if } k < i \\ k+1, & \text{if } k \geqslant i \end{cases}$$

$$\textit{codegeneracy maps} \qquad \forall n \geqslant 0\colon [n+1] \overset{s^i}{\twoheadrightarrow} [n] \qquad s^i(k) = \begin{cases} k, & \text{if } k \leqslant i \\ k-1, & \text{if } k > i \end{cases}$$

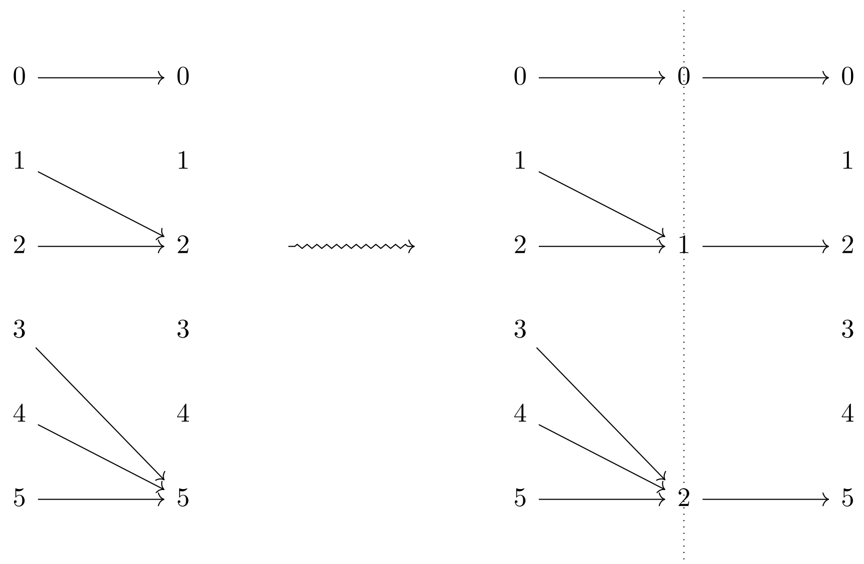
and these maps give rise to the following:

Lemma 2.3. *Any morphism $f\colon [n] \to [m]$ in Δ can be written as a (unique) composition*

$$f = d^{i_1} \circ \dots d^{i_r} \circ s^{j_1} \circ \dots s^{j_l}$$

with $0 \leqslant i_r < \dots < i_1 \leqslant m$ and $0 \leqslant j_1 \leqslant \dots < j_l < n$, where $r - l = m - n$.

The above lemma can be understood in an intuitive fashion by staring at



Indeed, the LHS above is a sample arrow $[5] \to [5]$ and the RHS gives a decomposition of this arrow into a composition of codegeneracy maps (the stuff to the left of the dotted line) followed by a composition of coface maps (the stuff to the right of the dotted line). Seeing that the first half of the RHS is a composition of codeneracy maps is done as follows: We start with the arrow $[5] \xrightarrow{s^1} [4]$ (which doubles up at 1), then postcompose this with $[4] \xrightarrow{s^2} [3]$ and this in turn we postcompose with $[3] \xrightarrow{s^2} [2]$. The complete composition $s^2 s^2 s^1$ is exactly equal to the morphism given by everything left to the dotted line. Analogously, the morphism given by everything right to the dotted line is precisely the composition $d^5 d^3 d^1$.
For the notion of a simplicial set, the above Lemma tells us that what a simplicial set X does to morphisms $f \in \Delta$ is completely determined by what it does to codegeneracy and coface maps. Hence the following definition makes sense:

Definition 2.4. Let $X \in$ sSet.

(*i*) The i-th *face operator* associated with the simplicial set X is given by

$$d_i := X_{d^i} : X_n \to X_{n-1}$$

(*ii*) The i-th *degeneracy operator* associated with the simplicial set X is given by

$$s_i := X_{s^i} : X_n \to X_{n+1}$$

(*iii*) An element $x \in X_n$ is called *non-degenerate* if $x \notin \bigcup_{i=0}^{n-1} s_i(X_{n-1})$.

Definition 2.5. Let $X, X' \in \text{sSet}$. The *product* $X \times X' \in \text{sSet}$ is the simplicial set given by $(X \times X')_n = X_n \times X'_n$ and $(X \times X')_f = X_f \times X'_f$ for all objects $[n]$ and all morphisms f in Δ.

Definition 2.6. Let $X, Y \in \text{sSet}$. We write $Y \subset X$ and say Y is a *simplicial subset* of X, if there is a monomorphism $Y \hookrightarrow X$.

Remark 2.7. More concretely, $Y \subset X$ if $Y_n \subset X_n$ for all $[n] \in \Delta$ and

$$X_f|_{Y_{\text{cod} f}} = Y_f$$

for all morphisms $f \in \Delta$.

Recall that the standard n-simplex Δ^n was defined by $\Delta(-, [n]) \in \text{sSet}$. There is a canonical way to extract important simplicial subsets of Δ^n:

Definition 2.8. Let $\mathscr{J}$ be a subset of the power set $\mathscr{P}([n])$ of the finite ordinal $[n]$ and define the simplicial subset

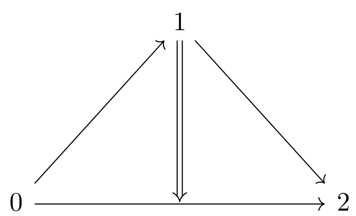
$$\Delta\langle \mathscr{J} \rangle \subset \Delta^n$$

by

$$\Delta\langle \mathscr{J} \rangle_m := \Big\{ f \in \Delta^n_m \mid \exists J \in \mathscr{J} : f([m]) \subset J \Big\}$$

Example 2.9. We list some of the most important simplicial subsets of Δ^n:

- The *standard* n*-simplex* itself is given by $\Delta^n := \Delta(-, [n]) = \Delta\langle \mathscr{P}([n]) \rangle \in$ sSet.
  - For $n = 2$ we have
    $\mathscr{P}([2]) = \Big\{ \varnothing, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\} \Big\}$
    We may then visualize $\Delta^2 = \Delta\langle \mathscr{P}([2]) \rangle$ as follows:

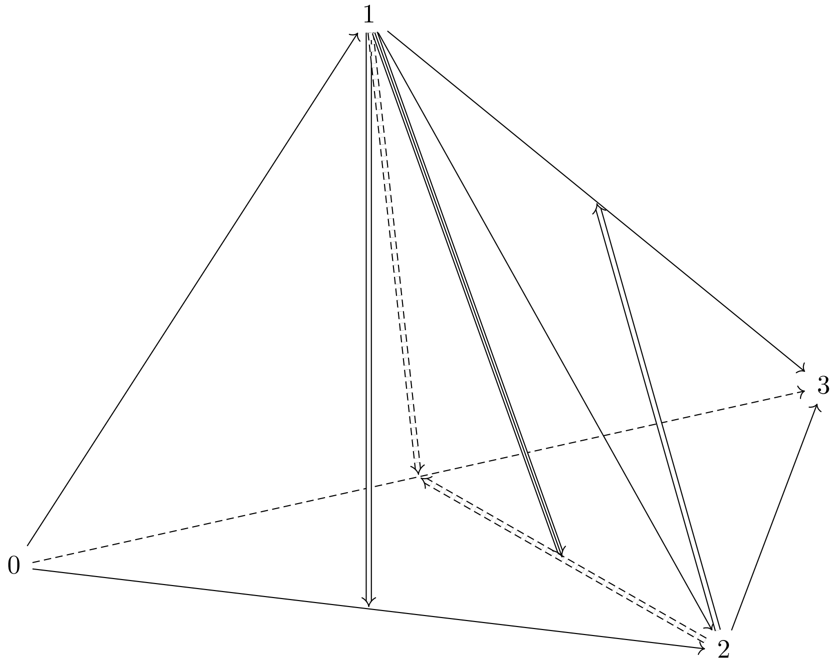


    The vertices of our (filled) triangle are represented by the singletons $0, 1, 2$, while the edges are given by the 2-element sets in $\mathscr{P}([2])$. The triple $\{0, 1, 2\}$ represents the 2-lined arrow going from the composite of $0 \to 1$ and $1 \to 2$ to the bottom $0 \to 2$. We think of Δ^2 as being the whole triangle (with area).

– For $n = 3$ we have

$$\mathscr{P}([3]) = \Big\{\varnothing, \{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{0,1,2\}, \{0,1,3\}, \{0,2,3\}, \{1,2,3\}, \{0,1,2,3\}\Big\}$$

Visualizing Δ^3 is then a little harder:



The above picture is to be interpreted analogously. The vertices are the singletons, the edges are given by the 2-element sets and the faces (the sides of our pyramid) are given by the triples in $\mathscr{P}([3])$ while the 3-lined arrow represents the filling $\{0, 1, 2, 3\}$.

- Consider the subset $\partial_i := \{0, \ldots, \widehat{i}, \ldots, n\} \subset [n]$ along with the induced collection of subsets

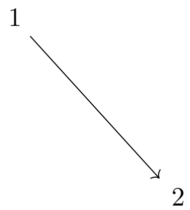
$$\mathscr{J}_i := \mathscr{P}([n]) \backslash \Big\{[n], \partial_0, \ldots, \widehat{\partial_i}, \ldots, \partial_n\Big\}$$

The *i-th face* of Δ^n is the simplicial set $\partial_i \Delta^n := \Delta\langle \mathscr{J}_i \rangle$.

– For $n = 2$ and $i = 0$ we have:

$$\mathscr{J}_0 = \Big\{\varnothing, \{0\}, \{1\}, \{2\}, \{1,2\}\Big\}$$

We then get a picture:

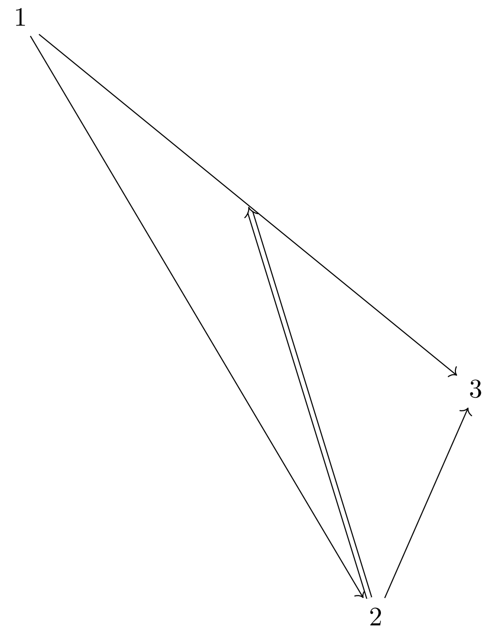


The vertices $\{1\}$ and $\{2\}$ are depicted in our diagram since they have a corresponding connecting edge $\{1, 2\}$. The singleton $\{0\}$, on the hand, is not connected to the above one-arrow graph, and therefore is not pictured at all.

– For $n = 3$ and $i = 0$ we have:

$$\mathscr{J}_0 = \Big\{\varnothing, \{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\Big\}$$

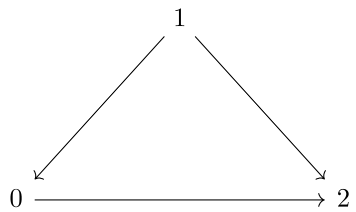
We therefore get the following picture:



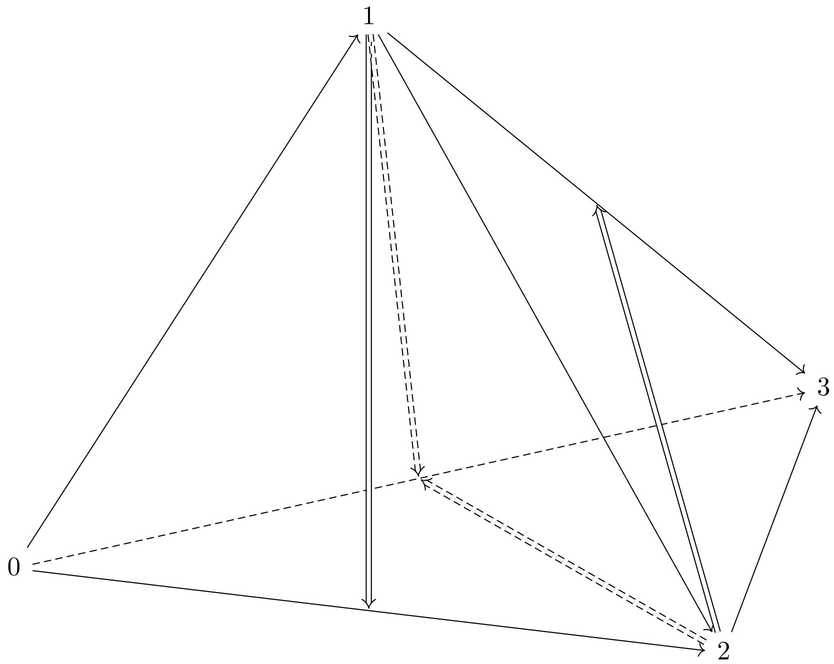
Again extending on the previous ideas, we really only draw the edges which happen to be connected by some face.

– Let $\mathscr{J} := \mathscr{P}([n])\backslash\{[n]\}$. The *simplicial boundary* of Δ^n is the simplicial subset $\partial\Delta^n := \Delta\langle\mathscr{J}\rangle$, i.e., $\partial\Delta^n = \bigcup \partial_i \Delta^n$

– For $n = 2$ we have the following picture:

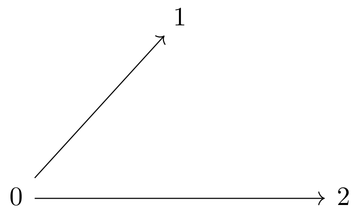


– For $n = 3$ we have the following picture:

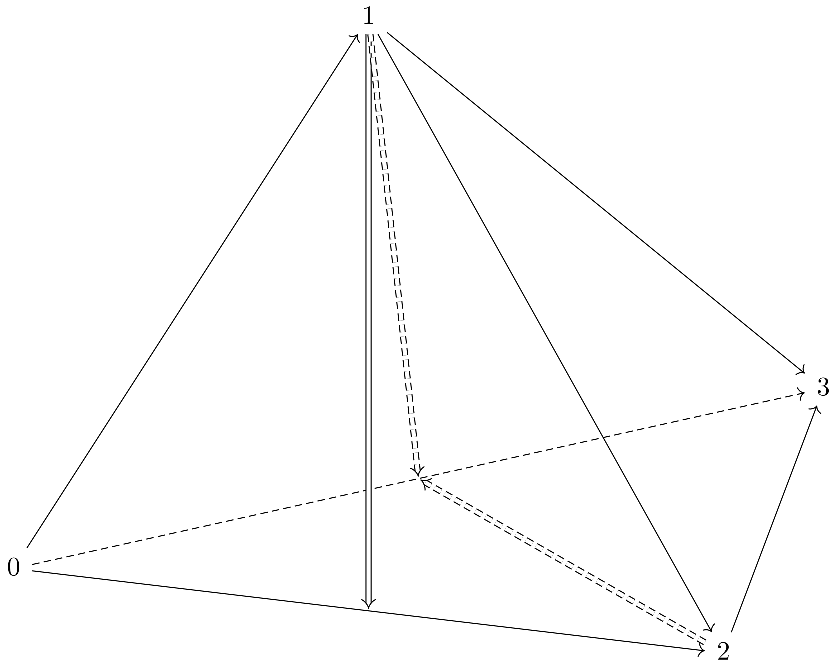


Hence all that is missing compared to the visualization of Δ^3 is the volume of the pyramid, that is, the squiggly arrow.

- Let $\mathscr{J} := \mathscr{P}([n]) \backslash \{[n], \partial_i\}$. The *i-th simplicial horn* of Δ^n is the simplicial subset $\Lambda_i^n := \Delta\langle \mathscr{J} \rangle$.
  – For $n = 2$ and $i = 0$, we obtain the picture:



– For $n = 3$ and $i = 0$ we have the picture:



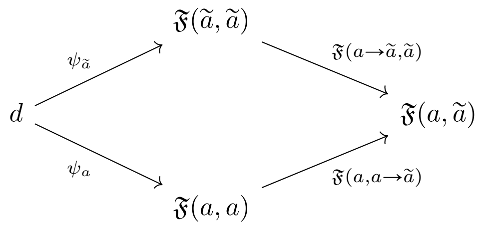
Looking at the 3-dimensional case, it is clear why this simplicial subset is called horn.

2.2. **Ends and Coends.** This chapter is based on the corresponding chapters in [7] as well as [23].

In the following we shall explain the notion of *ends* and *coends*. These will be very helpful machinery for what is to come.

Definition 2.10. Let $\mathfrak{F}\colon \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathscr{D}$ be a functor.

- A *wedge* for $\mathfrak{F}$ is a pair
$$\Big(d \in \mathscr{D}, \qquad \psi = \Big\{\psi_a\colon d \to \mathfrak{F}(a,a)\Big\}_{a\in\mathscr{A}}\Big)$$
such that for all morphisms $a \to \widetilde{a}$ in $\mathscr{A}$ we have a commutative diagram



For a wedge as above, the family of morphisms ψ will usually be denoted by $\psi\colon d \dot{\to} \mathfrak{F}$.
- A *cowedge* for $\mathfrak{F}$ is a pair
$$\Big(d \in \mathscr{D}, \qquad \varphi = \Big\{\varphi_a\colon \mathfrak{F}(a,a) \to d\Big\}_{a\in\mathscr{A}}\Big)$$

such that for all morphisms $a \to \widetilde{a}$ in $\mathscr{A}$ we have a commutative diagram

$$\begin{array}{ccccc} & & \mathfrak{F}(\widetilde{a}, \widetilde{a}) & & \\ & \overset{\mathfrak{F}(a, a\to\widetilde{a})}{\nearrow} & & \overset{\varphi_{\widetilde{a}}}{\searrow} & \\ \mathfrak{F}(\widetilde{a}, a) & & & & d \\ & \underset{\mathfrak{F}(a\to\widetilde{a}, \widetilde{a})}{\searrow} & & \underset{\varphi_a}{\nearrow} & \\ & & \mathfrak{F}(a, a) & & \end{array}$$

For a cowedge as above, the family of morphisms ψ will usually be denoted by $\psi\colon \mathfrak{F} \overset{\cdot\cdot}{\to} d$.

- An *end* of $\mathfrak{F}$ is a universal wedge
$$\Big(\int_{\mathscr{A}} \mathfrak{F} \in \mathscr{D}, \qquad \psi\colon \int_{\mathscr{A}} \mathfrak{F} \dot{\to} \mathfrak{F} \Big)$$
for $\mathfrak{F}$.
- A *coend* of $\mathfrak{F}$ is a universal cowedge
$$\Big(\int^{\mathscr{A}} \mathfrak{F} \in \mathscr{D}, \qquad \varphi\colon \mathfrak{F} \overset{\cdot\cdot}{\to} \int^{\mathscr{A}} \mathfrak{F} \Big)$$
for $\mathfrak{F}$.

More concretely, what does it mean to be a universal (co)wedge for $\mathfrak{F}$? Let us start off with ends: First of all note that the definition of a wedge induces a functor

$$E_{\mathfrak{F}}\colon \mathscr{D}^{\mathrm{op}} \to \mathrm{Set}, \qquad d \mapsto \Big\{ \text{wedges } \tau\colon d \dot{\to} \mathfrak{F} \Big\} \tag{2}$$

An end of $\mathfrak{F}$ is then defined to be a representation of the functor $E_{\mathfrak{F}}$, i.e. there is an object $\int_{\mathscr{A}} \mathfrak{F} \in \mathscr{D}$ such that

$$E_{\mathfrak{F}} \cong \mathscr{D}\Big(-, \int_{\mathscr{A}} \mathfrak{F} \Big)$$

By the Yoneda lemma this datum boils down to the statement that an end is a terminal object

$$\Big(\int_{\mathscr{A}} \mathfrak{F}, \qquad \zeta\colon \int_{\mathscr{A}} \mathfrak{F} \dot{\to} \mathfrak{F} \Big) \in \mathbf{el}(E_{\mathfrak{F}})$$

in the category of elements (see Remark 2.11) of $E_{\mathfrak{F}}$.

Remark 2.11. Recall that, in general, the category of elements $\mathbf{el}(\mathfrak{U})$ of a functor $\mathfrak{U}\colon \mathscr{C} \to \mathrm{Set}$ has as objects pairs $\Big(c \in \mathscr{C}, d \in \mathfrak{U}c \Big)$ and morphisms

$$\Big(c \in \mathscr{C}, d \in \mathfrak{U}c \Big) \longrightarrow \Big(\widetilde{c} \in \mathscr{C}, \widetilde{d} \in \mathfrak{U}\widetilde{c} \Big)$$

are morphisms $f\colon c \to \widetilde{c}$ in $\mathscr{C}$ such that $(\mathfrak{U}f)(d) = \widetilde{d}$.

Analogously, the definition of a cowedge induces a functor

$$C_{\mathfrak{F}}\colon \mathscr{D} \to \mathrm{Set}, \qquad Y \mapsto \Big\{ \text{cowedges } \tau\colon \mathfrak{F} \overset{\cdot\cdot}{\to} d \Big\}$$

By the same reasoning as above, a coend is simply an initial object

$$\Big(\int^{\mathscr{A}} \mathfrak{F}, \qquad \zeta\colon \mathfrak{F} \overset{\cdot\cdot}{\to} \int^{\mathscr{A}} \mathfrak{F} \Big) \in \mathbf{el}(C_{\mathfrak{F}})$$

Notation 2.12. It is sometimes useful to write $\int\limits_{a\in\mathscr{A}} \mathfrak{F}(a,a)$ and $\int\limits^{a\in\mathscr{A}} \mathfrak{F}(a,a)$ instead of $\int\limits_{\mathscr{A}} \mathfrak{F}$ and $\int\limits^{\mathscr{A}} \mathfrak{F}$.

Example 2.13. Let $\mathfrak{F}\colon \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathrm{Set}$. Then we have

$$\int_{\mathscr{A}} \mathfrak{F} \cong \mathrm{Set}\Big(\{\star\}, \int_{\mathscr{A}} \mathfrak{F}\Big) \cong \Big\{\text{wedges } \{\star\} \dot{\to} \mathfrak{F}\Big\}$$

Note that a wedge $\{\star\}\dot{\to}\mathfrak{F}$ corresponds to a family of elements $\Big(\tau(a) \in \mathfrak{F}(a,a)\Big)_{a\in\mathscr{A}}$ such that for all morphisms $a \to \tilde{a}$ in $\mathscr{A}$ we have

$$\mathfrak{F}(a \to \tilde{a}, \tilde{a})(\tau(\tilde{a})) = \mathfrak{F}(a, a \to \tilde{a})(\tau(a))$$

Remark 2.14. The notions of end and coend are dual. Indeed, the coend of a functor $\mathfrak{F}\colon \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathscr{D}$ is simply the end of the induced functor $\mathfrak{F}\colon (\mathscr{A}^{\mathrm{op}})^{\mathrm{op}} \times \mathscr{A}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$. Therefore, we may restrict ourselves to investigating coends, knowing that any result that holds for coends can be dualized to yield a result for ends.

Theorem 2.15. *If $\mathscr{D}$ is cocomplete and $\mathfrak{F}\colon \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathscr{D}$ is a functor, then the coend $\int^{\mathscr{A}} \mathfrak{F}$ exists in $\mathscr{D}$. It is given by the coequalizer of two suitable morphisms*

$$\coprod_{(f\colon a\to\tilde{a})\in\mathscr{A}} \mathfrak{F}(\tilde{a}, a) \;\underset{\xi_\star}{\overset{\xi^\star}{\rightrightarrows}}\; \coprod_{a\in\mathscr{A}} \mathfrak{F}(a,a)$$

Proof. We define the morphisms $\xi^\star$ and $\xi_\star$ by means of the universal property of the coproduct:

$$\begin{array}{ccc} \coprod\limits_{(f\colon a\to\tilde{a})\in\mathscr{A}} \mathfrak{F}(\tilde{a},a) & \overset{\exists!\xi_\star}{\dashrightarrow} & \coprod\limits_{a\in\mathscr{A}} \mathfrak{F}(a,a) \\ \uparrow{\scriptstyle \iota_f} & & \uparrow{\scriptstyle \iota_{\tilde{a}}} \\ \mathfrak{F}(\tilde{a},a) & \xrightarrow[\mathfrak{F}(\tilde{a},f)]{} & \mathfrak{F}(\tilde{a},\tilde{a}) \end{array} \qquad \begin{array}{ccc} \coprod\limits_{(f\colon a\to\tilde{a})\in\mathscr{A}} \mathfrak{F}(\tilde{a},a) & \overset{\exists!\xi^\star}{\dashrightarrow} & \coprod\limits_{a\in\mathscr{A}} \mathfrak{F}(a,a) \\ \uparrow{\scriptstyle \iota_f} & & \uparrow{\scriptstyle \iota_{a}} \\ \mathfrak{F}(\tilde{a},a) & \xrightarrow[\mathfrak{F}(f,a)]{} & \mathfrak{F}(a,a) \end{array}$$

A morphism $\zeta\colon \coprod\limits_{a\in\mathscr{A}} \mathfrak{F}(a,a) \to d$ with $\zeta\xi_\star = \zeta\xi^\star$ is then naturally identified with a cowedge $\mathfrak{F}\dot{\to}d$. Indeed, the cowedge associated to the morphism ζ, also denoted by the letter ζ, is defined by

$$\zeta = (\zeta_a \coloneqq \zeta\iota_a\colon \mathfrak{F}(a,a) \to d)_{a\in\mathscr{A}}$$

It is then not hard to check that

$$\begin{array}{ccccc} & & \mathfrak{F}(a,a) & & \\ & \overset{\mathfrak{F}(f,a)}{\nearrow} & & \overset{\zeta_a}{\searrow} & \\ \mathfrak{F}(\tilde{a},a) & & & & d \\ & \underset{\mathfrak{F}(\tilde{a},f)}{\searrow} & & \underset{\zeta_{\tilde{a}}}{\nearrow} & \\ & & \mathfrak{F}(\tilde{a},\tilde{a}) & & \end{array}$$

and thus the coequalizer of $\zeta^\star$ and $\zeta_\star$ is a universal cowedge for $\mathfrak{F}$. □

Remark 2.16. If we have a (co)continuous functor $\mathfrak{L}\colon \mathscr{D} \to \mathscr{E}$, then $\mathfrak{L}$ preserves (co)ends:

$$\mathfrak{L}\Big(\int^{\mathscr{A}} \mathfrak{F}\Big) \cong \int^{\mathscr{A}} \mathfrak{L}\mathfrak{F}, \qquad \mathfrak{L}\Big(\int_{\mathscr{A}} \mathfrak{F}\Big) \cong \int_{\mathscr{A}} \mathfrak{L}\mathfrak{F}$$

In particular, we have

$$\mathscr{D}(d, \int_{a\in\mathscr{A}} \mathfrak{F}(a,a)) \cong \int_{a\in\mathscr{A}} \mathscr{D}(d, \mathfrak{F}(a,a))$$

$$\mathscr{D}(\int^{a\in\mathscr{A}} \mathfrak{F}(a,a), d) \cong \int_{a\in\mathscr{A}} \mathscr{D}(\mathfrak{F}(a,a), d)$$

It turns out that the converse of Theorem 2.15 holds:

Theorem 2.17. *Let $\mathfrak{F}\colon \mathscr{A} \to \mathscr{D}$ be a functor. Define a functor $\mathfrak{F}'\colon \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathscr{D}$ by*

$$\begin{aligned} \mathfrak{F}'(a, \tilde{a}) &:= \mathfrak{F}(\tilde{a}) \\ \mathfrak{F}'(a, \tilde{a} \to \overline{a}) &:= \mathfrak{F}(\tilde{a} \to \overline{a}) \\ \mathfrak{F}'(a \to \overline{a}, \tilde{a}) &:= 1_{\mathfrak{F}\tilde{a}} \end{aligned}$$

Then the colimit of $\mathfrak{F}$ is the coend of $\mathfrak{F}'$:

$$\operatorname*{colim}_{a\in\mathscr{A}} \mathfrak{F}(a) \cong \int^{a\in\mathscr{A}} \mathfrak{F}'(a,a) = \int^{a\in\mathscr{A}} \mathfrak{F}a$$

Proof. A cowedge $\zeta\colon \mathfrak{F}' \ddot{\to} Y$ is a family of morphisms $\zeta_a\colon \mathfrak{F}(a) \to d$ such that for all $a \to \tilde{a}$ we have a commutative square

$$\begin{array}{ccc} \mathfrak{F}a & = & \mathfrak{F}a \\ {\scriptstyle \mathfrak{F}(a\to\tilde{a})}\downarrow & & \downarrow{\scriptstyle \zeta_a} \\ \mathfrak{F}\tilde{a} & \xrightarrow[\zeta_{\tilde{a}}]{} & d \end{array}$$

which is exactly a cocone for $\mathfrak{F}$. □

Lemma 2.18. *Let $\mathfrak{F}\colon \mathscr{A} \to \mathscr{D}$ and $\mathfrak{U}\colon \mathscr{A} \to \mathscr{D}$ be functors, and consider the induced Hom functor*

$$\mathscr{D}(\mathfrak{F}, \mathfrak{U})\colon \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathrm{Set}, \qquad (a, \tilde{a}) \mapsto \mathscr{D}(\mathfrak{F}a, \mathfrak{U}\tilde{a})$$

The end of this functor is given by

$$\mathscr{D}^{\mathscr{A}}(\mathfrak{F}, \mathfrak{U}) = \int_{a\in\mathscr{A}} \mathscr{D}(\mathfrak{F}a, \mathfrak{U}a)$$

Proof. From Example 2.13 we know that

$$\int_{a\in\mathscr{A}} \mathscr{D}(\mathfrak{F}(a), \mathfrak{U}(a)) = \left\{ \text{wedges } \{\star\} \dot{\to} \mathscr{D}(\mathfrak{F}, \mathfrak{U}) \right\}$$

The right hand side exactly corresponds to the set of natural transformations $\mathfrak{F} \to \mathfrak{U}$. □

Proposition 2.19 (Fubini for Coends)**.** *Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be categories and let $\mathfrak{F}\colon (\mathscr{A} \times \mathscr{B})^{\mathrm{op}} \times (\mathscr{A} \times \mathscr{B}) \to \mathscr{C}$. Then*

$$\int^{a\in\mathscr{A}} \int^{b\in\mathscr{B}} \mathfrak{F}(a,b,a,b) \cong \int^{(a,b)\in\mathscr{A}\times\mathscr{B}} \mathfrak{F}(a,b,a,b) \cong \int^{b\in\mathscr{B}} \int^{a\in\mathscr{A}} \mathfrak{F}(a,b,a,b)$$

where the above is to be understood in the sense that if one of these coends exists, then all of them exist and are isomorphic.

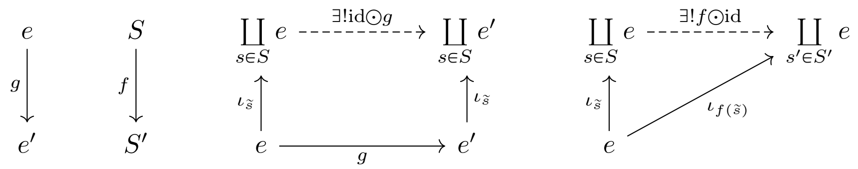
Proof. This immediately follows from the analogous result on colimits, see e.g. [35]. □

Coends turn out to be exceptionally useful for many reasons. One such reason is that coends can be used to decompose functors. In order to derive such a result we need to introduce some machinery first:

Definition 2.20. Let $\mathscr{D}$ be a cocomplete category. The bifunctor $\odot\colon \mathrm{Set} \times \mathscr{D} \to \mathscr{D}$ is given as follows:

$$S \odot d := \coprod_{s \in S} d \qquad (S, e) \in \mathrm{Set} \times \mathscr{D}$$

For morphisms in $\mathrm{Set} \times \mathscr{D}$ it is sufficient to define $\odot$ solely on morphisms of the kind (f, id) and (id, g) (by functoriality). This is done by means of the universal property of the coproduct:



where $\widetilde{s}$ runs over the set S.

Remark 2.21. The functor $\odot$ is a *copower* (or *tensoring*) of $\mathscr{D}$ over Set: This means that the functor

$$\odot\colon \mathrm{Set} \times \mathscr{D} \to \mathscr{D}$$

gives rise to natural isomorphisms

$$\mathscr{D}(S \odot d_1, d_2) \cong \mathrm{Set}(S, \mathscr{D}(d_1, d_2))$$

for all $S \in \mathrm{Set}$ and $d_1, d_2 \in \mathscr{D}$. This follows immediately from

$$\mathscr{D}\Big(S \odot d_1, d_2\Big) = \mathscr{D}\Big(\coprod_{s \in S} d_1, d\Big) \cong \prod_{s \in S} \mathscr{D}(d_1, d_2) \cong \mathrm{Set}\Big(S, \mathscr{D}(d_1, d_2)\Big)$$

Recall from analysis that if $\delta(x, y) := \delta(x - y)$ is the shifted Dirac-δ distribution, then any test function f can be written as

$$f = \int \delta(x, -) f(x) dx = \int \delta(-, y) f(y) dy$$

The aforementioned decomposition theorem for functions by means of the Dirac δ-distribution has a categorical analogue in terms of coends:

Theorem 2.22. *Let $\mathfrak{F}\colon \mathscr{A} \to \mathscr{D}$ and $\mathfrak{U}\colon \mathscr{A}^{\mathrm{op}} \to \mathscr{D}$ be functors, where $\mathscr{A}$ is a small category and $\mathscr{D}$ is cocomplete. Then we have the following natural isomorphisms:*

$$\mathfrak{F} \cong \int^{a \in \mathscr{A}} \text{よ} a \odot \mathfrak{F} a$$

$$\mathfrak{U} \cong \int^{a \in \mathscr{A}} \text{よ} a \odot \mathfrak{U} a$$

where よ, よ were introduced in .

Remark 2.23. The isomorphisms above are to be understood in the functor categories $\mathscr{D}^{\mathscr{A}}$ and $\mathscr{D}^{\mathscr{A}^{\mathrm{op}}}$, respectively. Taking the first of the two identities, the functor $\int^{a \in \mathscr{A}} \text{よ}_{\mathscr{A}} a \odot \mathfrak{F} a$ takes an object $\widetilde{a}$ in $\mathscr{A}$ and maps it to the coend

$$\int^{a \in \mathscr{A}} \mathscr{A}(a, \widetilde{a}) \odot \mathfrak{F} a$$

in the usual sense. Morphisms $\widetilde{a} \to \overline{a}$ are mapped to morphisms between the respective coends, and these are defined by means of the universal property of the coend.

Proof of Theorem 2.22. Let $d \in \mathscr{D}$ be arbitrary. We have natural isomorphisms

$$\begin{aligned}\mathscr{D}\Big(\int^{a\in\mathscr{A}} \mathscr{A}(a,\widetilde{a}) \odot \mathfrak{F}a, d\Big) &\cong \int_{a\in\mathscr{A}} \mathscr{D}(\mathscr{A}(a,\widetilde{a}) \odot \mathfrak{F}a, d) \cong \int_{a\in\mathscr{A}} \mathrm{Set}(\mathscr{A}(a,\widetilde{a}), \mathscr{D}(\mathfrak{F}a, d)) \\ &\cong \mathrm{Set}^{\mathscr{A}^{\mathrm{op}}}(\text{よ}_{\mathscr{A}}\widetilde{a}, \mathscr{D}(\mathfrak{F}, d)) \cong \mathscr{D}(\mathfrak{F}\widetilde{a}, d)\end{aligned}$$

where we have used Lemma 2.18 and the Yoneda lemma for the last two isomorphisms. Since $d \in \mathscr{D}$ was arbitrary the claim follows. □

Remark 2.24. One can dualize the above decomposition theorem as follows: Instead of the bifunctor $\odot$ one defines a bifunctor $\pitchfork\colon \mathrm{Set}^{\mathrm{op}} \times \mathscr{D} \to \mathscr{D}$ for a complete category $\mathscr{D}$ by

$$\mathrm{Set}^{\mathrm{op}} \times \mathscr{D} \ni (S, d) \mapsto S \pitchfork d := \prod_{s\in S} d \in \mathscr{D}$$

How this functor acts on morphisms is defined in the same way as for $\odot$ (now utilizing the universal property of the product). The functor $\pitchfork$ is a *power* (or *cotensor*) of $\mathscr{D}$ over Set: This means that for each $S \in \mathrm{Set}$ there are natural isomorphisms

$$\mathrm{Set}(S, \mathscr{D}(d_1, d_2)) \cong \mathscr{D}(d_1, S \pitchfork d_2)$$

The decomposition theorem for ends then reads

$$\mathfrak{F} \cong \int_{a\in\mathscr{A}} \text{よ}_{\mathscr{A}} a \pitchfork \mathfrak{F}a$$

for a functor $\mathfrak{F}\colon \mathscr{A} \to \mathscr{D}$.

Theorem 2.22 is sometimes also called the *density theorem*. This name is very much suiting since, if we are given a presheaf $\mathfrak{F} \in \mathrm{Set}^{\mathscr{A}^{\mathrm{op}}}$, then

$$\mathfrak{F} \cong \int^{a\in\mathscr{A}} \mathscr{A}(-,a) \odot \mathfrak{F}a \cong \operatorname*{colim}_{(a,x)\in\mathbf{el}(\mathfrak{F})} \mathscr{A}(-,a)$$

which tells us that the collection of representable presheaves in $\mathrm{Set}^{\mathscr{A}^{\mathrm{op}}}$ is dense in the category of presheaves $\mathrm{Set}^{\mathscr{A}^{\mathrm{op}}}$ (any presheaf is a colimit of representable presheaves). This will be explained (and proved) in more detail in Corollary 3.9.

Example 2.25. Recall that a simplicial set X is just a presheaf $\Delta^{\mathrm{op}} \to \mathrm{Set}$ on the simplex category. Thus Theorem 2.22 implies

$$X \cong \int^{[n]\in\Delta} \Delta^n \odot X_n \cong \operatorname*{colim}_{\mathbf{el}X} \Delta^n$$

Remark 2.26. Taking the coend of a functor in $\mathscr{D}^{\mathscr{A}^{\mathrm{op}}\times\mathscr{A}}$ may very well be interpreted as a functor

$$\int^{\mathscr{A}} \colon \mathscr{D}^{\mathscr{A}^{\mathrm{op}}\times\mathscr{A}} \to \mathscr{D}$$

The functor $\int^{\mathscr{A}}$ then fits into an adjunction

$$\mathscr{D}^{\mathscr{A}^{\mathrm{op}}\times\mathscr{A}} \underset{\mathscr{A}(-,-)\pitchfork -}{\overset{\int^{\mathscr{A}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathscr{D}$$

where $\mathscr{A}(-,-) \pitchfork -$ is the functor which takes an object $d \in \mathscr{D}$ to the functor

$$\prod_{W(-,-)} d \in \mathscr{D}^{\mathscr{A}\times\mathscr{A}^{\mathrm{op}}}$$

In order to see that this adjunction holds, we first note that the bifunctor $\mathscr{A}(-,-)$ is equal to the coend

$$\int^{a\in\mathscr{A}^{\mathrm{op}}} よ(a,a)$$

where $よ: \mathscr{A} \times \mathscr{A}^{\mathrm{op}} \to \mathrm{Set}^{(\mathscr{A}\times\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}}$ is the Yoneda embedding. Indeed, a quick application of the coend calculus yields

$$\begin{aligned}\Big(\int^{a} よ(a,a)\Big)(a',a'') &= \int^{a} \mathscr{A}(a',a) \times \mathscr{A}(a,a'') \\ &\cong \mathscr{A}(a',a'')\end{aligned}$$

by the fact that $\odot = \times$ in Set and Theorem 2.22. In order to then get the adjunction

$$\mathscr{D}(\int^{\mathscr{A}} \mathfrak{F}, d) \cong \mathrm{Set}^{\mathscr{A}^{op}\times\mathscr{A}}(W, \mathscr{D}(\mathfrak{F}, d))$$

we calculate

$$\begin{aligned}\mathscr{D}(\int^{\mathscr{A}} \mathfrak{F}, d) &\cong \int_{a} \mathscr{D}(\mathfrak{F}(a,a), d) \\ &\cong \int_{a} \mathrm{Set}^{\mathscr{A}^{\mathrm{op}}\times\mathscr{A}}(よ(a,a), \mathscr{D}(\mathfrak{F}, d)) \\ &\cong \mathrm{Set}^{\mathscr{A}^{\mathrm{op}}\times\mathscr{A}}(\mathscr{A}(-,-), \mathscr{D}(\mathfrak{F}, d)) \\ &\cong \int_{a,a'} \mathrm{Set}(\mathscr{A}(a,a'), \mathscr{D}(\mathfrak{F}(a,a'), d)) \\ &\cong \int_{a,a'} \mathscr{D}(\mathfrak{F}(a,a'), \mathscr{A}(a,a') \pitchfork d) \\ &\cong \mathscr{D}^{\mathscr{A}^{\mathrm{op}}\times\mathscr{A}}(\mathfrak{F}, \mathscr{A}(-,-) \pitchfork d)\end{aligned}$$

2.3. **Nerve Realization Adjunction.** This chapter is mostly based on the Nlab article nerve and realization, [28] and [33].

Definition 2.27. Let $\mathfrak{F}: \mathscr{C} \to \mathscr{D}$ be a functor, for $\mathscr{C}$ a small category. Then the *nerve functor associated to* $\mathfrak{F}$ is the functor

$$\mathfrak{N}_{\mathfrak{F}}: \mathscr{D} \to \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}, \qquad d \mapsto \mathscr{D}(\mathfrak{F}, d)$$

It turns out that the previous construction is of particular interest to us: The functor $\mathfrak{N}_{\mathfrak{F}}$ will have a left adjoint $|-|_{\mathfrak{F}}$ in most practical cases, and the corresponding adjunction will be used to define several important future notions. The question now is what kind of assumptions we have to impose on the ingredients $\mathfrak{F}, \mathscr{C}$ and $\mathscr{D}$ so that $\mathfrak{N}_{\mathfrak{F}}$ indeed admits a left adjoin. It turns out that a sufficient condition is to assume cocompleteness of $\mathscr{D}$:

Theorem 2.28. *Let $\mathscr{C}$ be a small category and let $\mathscr{D}$ be a cocomplete category. If $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ is a functor, then the induced nerve functor $\mathfrak{N}_{\mathfrak{F}}\colon \mathscr{D} \to \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$ has a left adjoint*

$$|-|_{\mathfrak{F}}\colon \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathscr{D}\colon \mathfrak{N}_{\mathfrak{F}}$$

given on objects $\mathfrak{U} \in \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$ by the coend

$$|\mathfrak{U}|_{\mathfrak{F}} := \int^{c\in\mathscr{C}} \mathfrak{U}c \odot \mathfrak{F}c$$

In particular, $|-|_{\mathfrak{F}}$ is the unique cocontinuous extension of $\mathfrak{F}$, that is,

$$|\text{よ} c|_{\mathfrak{F}} \cong \mathfrak{F}c$$

for all objects $c \in \mathscr{C}$.

Proof of Theorem 2.28. By Theorem 2.22

$$|\text{よ} c|_{\mathfrak{F}} = \int^{\tilde{c}\in\mathscr{C}} \mathscr{C}(\tilde{c}, c) \odot \mathfrak{F}\tilde{c} \cong \mathfrak{F}c$$

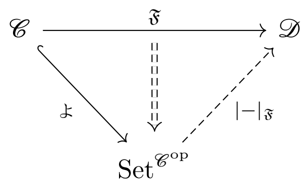
For the adjoint correspondence we calculate:

$$\mathscr{D}(\int^{c\in\mathscr{C}} \mathfrak{U}c \odot \mathfrak{F}c, d) \cong \int_{c\in\mathscr{C}} \mathscr{D}(\mathfrak{U}c \odot \mathfrak{F}c, d) \cong \int_{c\in\mathscr{C}} \mathrm{Set}(\mathfrak{U}c, \mathscr{D}(\mathfrak{F}c, d)) \cong \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}(\mathfrak{U}, \mathfrak{N}_{\mathfrak{F}}d)$$

□

Remark 2.29. Some remarks for generalizations and important notions are in order:

- The functor $|-|_{\mathfrak{F}}$ acts on morphisms $(\mathfrak{U} \to \mathfrak{H})$ in $\mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$ by means of the universal property of the coend.
- $|-|_{\mathfrak{F}}$ is the left Kan extension of $\mathfrak{F}$ along the Yoneda embedding よ:



  This will be explained, in detail, later in section 3.
- There is a dual Theorem to Theorem 2.28: Let $\mathscr{D}$ be a complete category and consider a functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$. From this we may define the dual nerve (or co-nerve)

$$\widehat{\mathfrak{N}}_{\mathfrak{F}}\colon \mathscr{D}^{\mathrm{op}} \to \mathrm{Set}^{\mathscr{C}}, \qquad d \mapsto \mathscr{D}(d, \mathfrak{F})$$

  The functor $\widehat{\mathfrak{N}}_{\mathfrak{F}}$ then fits into an adjunction

$$\mathscr{C}^{\mathrm{op}} \underset{\overline{\underline{\;-\;}}_{\mathfrak{F}}}{\overset{\widehat{\mathfrak{N}}_{\mathfrak{F}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Set}^{\mathscr{S}}$$

  where the left adjoint $\overline{\underline{\;-\;}}_{\mathfrak{F}}$ is given on objects $\mathfrak{U} \in \mathrm{Set}^{\mathscr{C}}$ by the end

$$\overline{\underline{\mathfrak{U}}}_{\mathfrak{F}} := \int_{c\in\mathscr{C}} \mathfrak{U}c \pitchfork \mathfrak{F}c$$

  In order to see that this is truly a left adjoint to $\widehat{\mathfrak{N}}_{\mathfrak{F}}$ we calculate:

$$\mathscr{D}^{\mathrm{op}}(\overline{\underline{\mathfrak{U}}}_{\mathfrak{F}}, d) \cong \mathscr{D}(d, \overline{\underline{\mathfrak{U}}}_{\mathfrak{F}})$$

$$
\begin{aligned}
&\cong \int_{c \in \mathscr{C}} \mathscr{D}(d, \mathfrak{U}c \pitchfork \mathfrak{F}c) \\
&\cong \int_{c \in \mathscr{C}} \mathrm{Set}(\mathfrak{U}c, \mathscr{D}(d, \mathfrak{F}c)) \\
&\cong \mathrm{Set}^{\mathscr{C}}(\mathfrak{U}, \widehat{\mathfrak{N}}_{\mathfrak{F}} d)
\end{aligned}
$$

Moreover, $\overline{\underline{\;-\;}}_{\mathfrak{F}}$ is the unique continuous extension of $\mathfrak{F}$, that is,

$$\overline{\underline{\text{よ} c}}_{\mathfrak{F}} \cong \int_{\tilde{c} \in \mathscr{C}} \mathscr{C}(c, \tilde{c}) \pitchfork \mathfrak{F}\tilde{c} \cong \mathfrak{F}c$$

Example 2.30 (Geometric realization)**.** Recall from the very beginning of section 2 that there is a functor $|-|\colon \Delta \to \mathrm{Top}$ which sends $[n]$ to the standard topological n-simplex

$$|\Delta^n| = \Big\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geqslant 0, \sum_i t_i = 1\Big\}$$

This induces a functor

$$\Pi_{\leqslant\infty} := \mathfrak{N}_{|-|}\colon \mathrm{Top} \to \mathrm{sSet}, \qquad \Pi_{\leqslant\infty} Y := \mathrm{Top}(|-|, Y) \in \mathrm{sSet}$$

By Theorem 2.28 we therefore obtain a left adjoint for the *total singular complex* $\Pi_{\leqslant\infty}$ given by

$$|-|\colon \mathrm{sSet} \to \mathrm{Top}, \qquad |X| = \int^{[n] \in \Delta} X_n \odot |\Delta^n|$$

Unravelling the definition of the coend, $|X|$ is isomorphic to the quotient space

$$|X| = \Big(\coprod_{n \geqslant 0} X_n \times |\Delta^n|\Big) \Big/ \sim$$

where $\sim$ is the equivalence relation generated by pairs

$$\Big(X_f(x), y\Big) \sim \Big(x, |f|(y)\Big), \qquad (f, x, y) \in \Delta^n_m \times X_n \times |\Delta^n|$$

Since Δ has a generating set of morphisms, the above equivalence relation can also be merely stated in terms of face and degeneracy maps. Thus, face and degeneracy maps already provide all the necessary information for us to know how to glue.

Example 2.31. Let $\mathscr{C}$ be a small category and let

$$\iota\colon \Delta \to \mathrm{Cat}$$

be the inclusion functor of the simplex category into small categories, i.e., the object $[n]$ is mapped to the category $\iota[n] = \{0 \to 1 \to \ldots \to n\}$, which has $n+1$ objects and precisely n non-identity morphisms. An order preserving map $f\colon [n] \to [m]$ is mapped to the corresponding functor $\iota f\colon \iota[n] \to \iota[m]$ induced from f. The nerve of the category $\mathscr{C}$ is defined to be the simplicial set

$$\mathfrak{N}\mathscr{C} := \mathfrak{N}_\iota \mathscr{C}\colon \Delta^{\mathrm{op}} \to \mathrm{Set}, \qquad [n] \mapsto \mathrm{Cat}(\iota[n], \mathscr{C})$$

More concretely, if $\mathscr{C}$ is a category with $\mathscr{C}_0, \mathscr{C}_1$ the corresponding sets of objects and morphisms respectively, then $\mathfrak{N}\mathscr{C}$ is the simplicial set with simplices:

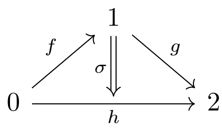
$$
\begin{aligned}
\mathfrak{N}\mathscr{C}_0 &= \mathscr{C}_0 \\
\mathfrak{N}\mathscr{C}_1 &= \mathscr{C}_1 \\
\mathfrak{N}\mathscr{C}_2 &= \{\text{pairs of composable morphisms in } \mathscr{C}\} = \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \\
&\;\;\vdots
\end{aligned}
$$

$$\mathfrak{N}\mathscr{C}_n = \{\text{strings of } n\text{-composable morphisms in } \mathscr{C}\} = \mathscr{C}_1 \times_{\mathscr{C}_0} \ldots \times_{\mathscr{C}_0} \mathscr{C}_1$$

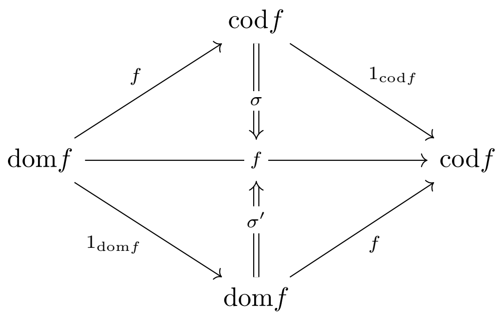
This leads to the so-called *nerve functor* $\mathfrak{N}\colon \mathrm{Cat} \to \mathrm{sSet}$ ($\mathfrak{N}\mathscr{C}$ is called the nerve of the category $\mathscr{C}$). Its left adjoint is called *first truncation* and is given again by means of Theorem 2.28:

$$\mathfrak{h}\colon \mathrm{sSet} \to \mathrm{Cat}, \qquad X \mapsto \int^{[n]\in\Delta} X_n \odot \iota[n]$$

The functor $\mathfrak{h}$ assigns to any simplicial set its corresponding *homotopy category* $\mathfrak{h}X$. This description of $\mathfrak{h}$ is fine, if we just want to know about the existence of the left adjoint. However, the coend formula above does not really offer insights as to what the category $\mathfrak{h}X$ is really all about. Hence we shall also present a different, more explicit construction of $\mathfrak{h}X$: We start off by defining the set of objects of $\mathfrak{h}X$ to be X_0. The set of morphisms for $\mathfrak{h}X$ is freely generated from X_1 subject to some relations given by elements in X_2 as follows: The degeneracy map $s_0\colon X_0 \to X_1$ picks out an identity morphism for every object $x \in X_0$, that is, $1_x := s_0(x) \in X_1$ for all $x \in X_0$. The face maps $d_1, d_0\colon X_1 \to X_0$ assign domain and codomain to arrows $f \in X_1$, that is, $\mathrm{dom} f := d_1(f)$ and $\mathrm{cod} f := d_0(f)$. To then obtain $\mathfrak{h}X$, we consider the free graph on X_0 generated by the arrows X_1 and then impose the relation $h = gf$ if there exists a 2-simplex $\sigma \in X_2$ such that $d_2\sigma = f, d_0\sigma = g$ and $d_1\sigma = h$. Representing this graphically we obtain



Composition in $\mathfrak{h}X$ is then associative (it is a free graph after all). Unitality is established as follows: For $f \in X_1$ we have to verify that there are 2-simplices $\sigma, \sigma' \in X_2$ so that



Verifying this e.g. for the upper triangle goes as follows: Define $\sigma := s_1(f)$. Then $d_2 s_1(f) = X_{s^1 d^2}(f) = f$ and $d_1 s_1(f) = X_{s^1 d^1}(f) = f$ since $s^1 d^1 = s^1 d^2 = \mathrm{id}$. In particular,

$$\begin{aligned} d_0 s_1(f) &= X_{s^1 d^0}(f) \\ &= X_{d^0 s^0}(f) \\ &= s_0(d_0(f)) \\ &= s_0(\mathrm{cod} f) \\ &= 1_{\mathrm{cod} f} \end{aligned}$$

This shows that $\mathfrak{h}X$ is indeed a category. For morphisms $X \to Y$ in sSet we realize that naturality gives rise to a functor $\mathfrak{h}X \to \mathfrak{h}Y$ and this is functorial. It remains to verify that $\mathfrak{h}$ is a left adjoint for $\mathfrak{N}$. However, we really only have to verify this for representable simplicial sets Δ^n, that is, we have to show

$$\mathrm{Cat}(\mathfrak{h}\Delta^n, \mathscr{C}) \cong \mathrm{sSet}(\Delta^n, \mathfrak{N}\mathscr{C})$$

However, by the Yoneda Lemma the RHS is simply $\mathfrak{N}\mathscr{C}_n$,i.e., the set of chains of n composable morphisms. Looking at the LHS we immediately realize that $\mathfrak{h}\Delta^n$ is isomorphic to the category $\iota[n]$, from which the adjunction follows. Since left adjoints are unique up to natural isomorphism, this verifies that both constructions for $\mathfrak{h}$ agree up to isomorphism.

Example 2.32. We construct a functor $\mathrm{sd}\colon \Delta \to \mathrm{sSet}$. In order to do so, we need some preliminary notions:

- Let $\mathrm{Poset}([n])$ denote the poset of nonempty subsets of the ordinal $[n]$, ordered by inclusion.
- For a morphism $f\colon [n] \to [m]$ in Δ we get a poset map

$$f_\star\colon \mathrm{Poset}([n]) \to \mathrm{Poset}(\Delta^m), \qquad f_\star(M) := f(M)$$

- This defines the poset functor $\mathrm{Poset}\colon \Delta \to$ **Poset**, where **Poset** denotes the category of posets.

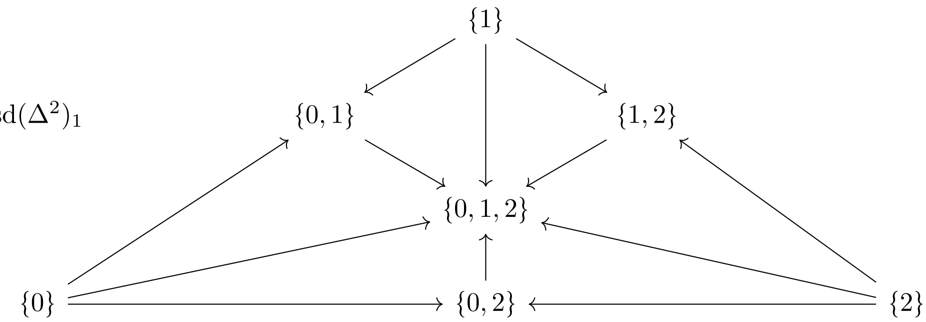
By means of the poset functor and the nerve functor $\mathfrak{N}$ from Example 2.31 we may define sd by

$$\mathrm{sd} := \mathfrak{N} \circ \mathrm{Poset}\colon \Delta \to \mathrm{sSet}, \qquad \mathrm{sd}(\Delta^n)_m = \mathrm{Cat}([m], \mathrm{Poset}([n]))$$

Applying the functor sd to Δ^1 and Δ^2 and looking at the respective 1-simplices yields the following picture:

$$\mathrm{sd}(\Delta^1)_1 \qquad \{0\} \longrightarrow \{0,1\} \longleftarrow \{1\}$$

$\mathrm{sd}(\Delta^2)_1$



From the preceding two diagrams it is not very surprising that the geometric realization $|\mathrm{sd}\Delta^n|$ is exactly the barycentric subdivision of $|\Delta^n|$. Having defined the functor sd, we may consider the corresponding nerve functor

$$\mathrm{Ex} := \mathfrak{N}_{\mathrm{sd}}\colon \mathrm{sSet} \to \mathrm{sSet}, \qquad X \mapsto \mathrm{sSet}(\mathrm{sd}, X)$$

By Theorem 2.28 we get a left adjoint to Ex, the unique cocontinuous extension $\mathrm{sd}\colon \mathrm{sSet} \to \mathrm{sSet}$ given by

$$\mathrm{sd}X = \int^{[n]\in\Delta} X_n \odot \mathrm{sd}\Delta^n$$

Example 2.33. Theorem 2.28 also proves that sSet is cartesian closed, i.e., for every simplicial set Y the functor $- \times Y\colon \mathrm{sSet} \to \mathrm{sSet}$ has a right adjoint. This right adjoint is referred to as the *internal hom* (this concept will be explained in section 4.2 in detail). In fact, this is true for all categories of set-valued presheaves endowed with the induced cartesian structure. Indeed, fix a simplicial set Y and let $\mathfrak{F}\colon \Delta \to \mathrm{sSet}$ be the functor

$$[n] \mapsto \Delta^n \times Y$$
$$f \mapsto よ_\Delta(f) \times 1_Y$$

The corresponding nerve to $\mathfrak{F}$ is then given by $\mathfrak{N}_{\mathfrak{F}} = \mathrm{sSet}(よ_\Delta \times Y, -)$. We define the internal hom as $Y^{(-)} := \mathfrak{N}_{\mathfrak{F}} = \mathrm{sSet}(よ_\Delta \times Y, -)$. It may then be checked that $|-|_{\mathfrak{F}} = - \times Y$ and therefore, by Theorem 2.28, we obtain the desired adjunction.

Example 2.34. Denote by $\Delta_{\leqslant n}$ the full subcategory of Δ with objects $[0], \dots, [n]$. The inclusion functor $\Delta_{\leqslant n} \overset{i}{\hookrightarrow} \Delta$ can be viewed as a functor $\mathrm{sk}_n\colon \Delta_{\leqslant n} \hookrightarrow \mathrm{sSet}$ by means of the Yoneda Lemma, and this functor induces the *truncation functor*

$$\mathrm{tr}_n\colon \mathrm{sSet} \to \mathrm{sSet}_{\leqslant n} := \mathrm{Set}^{\Delta^{\mathrm{op}}_{\leqslant n}}$$
$$X \mapsto \mathrm{sSet}(\mathrm{sk}_n, X) \cong X \circ i$$

By Theorem 2.28 tr_n has a left adjoint $\mathrm{sk}_n\colon \mathrm{sSet}_{\leqslant n} \to \mathrm{sSet}$, called the *n-skeleton*, given on objects $X \in \mathrm{sSet}_{\leqslant n}$ by

$$\mathrm{sk}_n X = \int^{[k]\in\Delta_{\leqslant n}} X_k \odot \mathrm{sk}_n[k]$$

We may even extend the domain of sk_n further by precomposing with tr_n:

$$\mathbf{sk}_n := \mathrm{sk}_n \circ \mathrm{tr}_n\colon \mathrm{sSet} \to \mathrm{sSet}$$

Example 2.35. Recall that a groupoid is a category for which all hom-sets only contain isomorphisms. The category of groupoids Grpd is the full subcategory in Cat which has as objects the collection of groupoids. It can be shown that Grpd is cocomplete. Therefore one may apply Theorem 2.28 to the functor $\leftrightarrow\colon \Delta \to \mathrm{Grpd}$ which takes $[n]$ and maps it to the groupoid $n^{\leftrightarrow} := \{0 \cong 1 \cong \dots \cong n\}$. The *fundamental groupoid functor* $\mathfrak{h}^{\leftrightarrow}\colon \mathrm{sSet} \to \mathrm{Grpd}$ is then defined as the realization $|-|_{\leftrightarrow}$ obtained by means of Theorem 2.28. More concretely, for all simplicial sets X we have

$$\mathfrak{h}^{\leftrightarrow} X = \int^{[n]\in\Delta} X_n \odot n^{\leftrightarrow}$$

An explicit construction is analogous to the one given in Example 2.31 with the sole difference of adding all the necessary inverses. If X is a topological space and $x \in X$ is a point, then we may first consider the total singular complex $\Pi_{\leqslant\infty} X = \mathrm{Top}(|-|, X) \in \mathrm{sSet}$. Applying the fundamental groupoid functor $\mathfrak{h}^{\leftrightarrow}$ yields a groupoid $\mathfrak{h}^{\leftrightarrow}(\Pi_{\leqslant\infty} X)$ whose objects are given by the points in X. Thus we may consider the hom-set

$$\pi_1(X, x) := \mathfrak{h}^{\leftrightarrow}(\Pi_{\leqslant\infty} X)(x, x)$$

which exactly recovers the fundamental group of a topological space X at the point $x \in X$.

2.4. **Homotopy Theory.** The following is based on [28].

When we think of homotopy theory, we think of topological spaces and continuous maps between these, along with the essential ingredient of the unit interval $I = [0,1]$. In fact, recalling the definitions, if f and g are continuous maps with the same domain and codomain (morphisms in the category Top), then a homotopy between f and g is a continuous map

$$h\colon [0,1] \times \mathrm{dom} f \longrightarrow \mathrm{cod} f$$

such that $h|_{\{0\}\times \mathrm{dom} f} \equiv f$ and $h|_{\{1\}\times \mathrm{dom} f} \equiv g$. A continuous map f is then said to be a homotopy equivalence, if f has an inverse up to homotopy, i.e., there is some composable morphism f' such that $f \circ f'$ and $f' \circ f$ are homotopic to the respective identities. The category of topological spaces is not the only category that gives rise to a homotopy theory, as the name of this section might have already implied. In this scheme of things, the role played by the standard 1-simplex Δ^1 in sSet will be analogous to the role played by the interval $[0,1] \cong |\Delta^1|$ in Top (concerning the definition of homotopies).

Definition 2.36. Let f, g be morphisms in sSet with the same domain and codomain.

- A *simplicial homotopy* between f and g is a simplicial map $h\colon \Delta^1 \times \mathrm{dom} f \to \mathrm{cod} f$ such that we have a commutative diagram

$$\begin{array}{ccccc}
\Delta^0 \times \mathrm{dom} f & \xrightarrow{\text{よ}_\Delta(d^1)\times 1_{\mathrm{dom} f}} & \Delta^1 \times \mathrm{dom} f & \xleftarrow{\text{よ}_\Delta(d^0)\times 1_{\mathrm{dom} f}} & \Delta^0 \times \mathrm{dom} f \\
\cong \big\uparrow & & \big\downarrow h & & \big\uparrow \cong \\
\mathrm{dom} f & \xrightarrow[g]{} & \mathrm{cod} f & \xleftarrow[f]{} & \mathrm{dom} f
\end{array}$$

- A *simplicial homotopy equivalence* is a simplicial map f for which there exists a (composable) simplicial map f' such that $f' \circ f$ and $f \circ f'$ are homotopic to the respective identities.

Therefore, doing homotopy theory is not something we can exclusively do in the category of topological spaces, but the intrinsic structure of sSet also allows for such a theory to be developed. In fact, many categories give rise to certain kinds of homotopy theories. These special kinds of categories, be it homotopical categories or model categories etc., will be covered in detail in Chapter 5.

Remark 2.37. Equivalently, a simplicial homotopy between $f, g\colon X \to Y$ is a map $h\colon X \to Y^{\Delta^1}$ such that

$$Y^{d^1} h = f, \qquad Y^{d^0} h = g$$

where Y^{d^j} are induced from the two maps $d^j\colon [0] \to [1]$ and the definition of the internal hom

$$Y^{\Delta^1} := \mathrm{sSet}(\text{よ} \times \Delta^1, Y) \in \mathrm{sSet}, \qquad [n] \mapsto \mathrm{sSet}(\Delta^n \times \Delta^1, Y)$$

See also Example 2.33 and Chapter 4.2 for more details.

Proposition 2.38. *The singular complex functor* $\Pi_{\leqslant\infty}\colon \mathrm{Top} \to \mathrm{sSet}$*, defined in Example 2.30, maps continuous homotopies to simplicial homotopies. In particular, continuous homotopy equivalences are mapped to simplicial homotopy equivalences.*

Proof. Let h be a continuous map $[0,1] \times X \longrightarrow Y$. Since $\Pi_{\leqslant\infty}$ is a right adjoint, it preserves limits and therefore

$$\Pi_{\leqslant\infty}([0,1] \times X) \cong \Pi_{\leqslant\infty}([0,1]) \times \Pi_{\leqslant\infty}(X)$$

We may thus write $\Pi_{\leqslant\infty}(h)$ as

$$\Pi_{\leqslant\infty}(h)\colon \Pi_{\leqslant\infty}([0,1]) \times \Pi_{\leqslant\infty}(X) \to \Pi_{\leqslant\infty}(Y)$$

Taking the adjunct of the canonical homeomorphism $|\Delta^1| \to [0,1]$ yields a simplicial map $\xi\colon \Delta^1 \to \Pi_{\leqslant\infty}([0,1])$. By precomposing with $\xi \times 1_{\Pi_{\leqslant\infty}(X)}$ we obtain a simplicial map

$$\Delta^1 \times \Pi_{\leqslant\infty}(X) \longrightarrow \Pi_{\leqslant\infty}(Y)$$

which is the desired simplicial homotopy. □

2.4.1. *Kan Complexes.* Recall the definition of the i-th horn Λ^n_i of the standard n-simplex from Example 2.9.

Definition 2.39. A simplicial set $X \in \mathrm{sSet}$ is called a *Kan complex* if it satisfies the following horn filling conditions: For all $0 \leqslant i \leqslant n, n > 0$, every map

$$\Lambda^n_i \xrightarrow{f} X$$

can be extended (this may be non-unique) to a commutative diagram

$$\begin{array}{ccc} \Lambda^n_i & \xrightarrow{f} & X \\ \big\downarrow & \nearrow_{\tilde{f}} & \\ \Delta^n & & \end{array}$$

Remark 2.40. Equivalently, the above horn filling conditions boil down to the statement that the corresponding inclusion maps $\Lambda^n_i \hookrightarrow \Delta^n$ induce surjections

$$\mathrm{sSet}(\Delta^n, X) \longrightarrow \mathrm{sSet}(\Lambda^n_i, X)$$

For later use we shall also define the more general notion of a *Kan fibration*:

Definition 2.41. A simplicial map $f \in \mathrm{sSet}$ is said to be a *Kan fibration*, if it has the right lifting property with respect to all horn inclusions, i.e., for each $1 \leqslant k \leqslant n$ the diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathrm{dom} f \\ \big\downarrow & \overset{\exists d}{\nearrow} & \big\downarrow f \\ \Delta^n & \longrightarrow & \mathrm{cod} f \end{array}$$

allows for a lift $d\colon \Delta^n \to \mathrm{dom} f$.

Remark 2.42. The notion of a Kan fibration is more general than that of a Kan complex. Indeed, X is a Kan complex if and only if the unique map $X \to \Delta^0$ is a Kan fibration.

It turns out that any topological space may be viewed as a Kan complex. More precisely:

Theorem 2.43. *The simplicial set $\Pi_{\leqslant\infty}(Y)$ is a Kan complex for all $Y \in \mathrm{Top}$.*

Proof. Fix $0 \leqslant i \leqslant n$ and consider the inclusion

$$|\Lambda^n_i| \underset{i}{\overset{r}{\leftrightarrows}} |\Delta^n|$$

which admits a retract $r\colon |\Delta^n| \to |\Lambda^n_i|$, i.e., $r \circ i = \mathrm{id}_{|\Lambda^n_i|}$. Explicitly, a retract $r\colon |\Delta^n| \to |\Lambda^n_i|$ may be given by

$$r(t_0, \ldots, t_n) \coloneqq (t_0 - c, \ldots, t_{i-1} - c, t_i + nc, t_{i+1} - c, \ldots, t_n - c)$$

where $c \coloneqq \min(t_0, \ldots, t_n)$ and $|\Lambda^n_i|$ is given by

$$\Big\{(t_0, \ldots, t_n) \mid t_j = 0 \text{ for some } j \neq i\Big\} \subset |\Delta^n|$$

Using the adjunction $|-| \dashv \Pi_{\leqslant\infty}$, the extension problem

$$\begin{array}{ccc} \Lambda^n_i & \xrightarrow{f} & \Pi_{\leqslant\infty} X \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

is then equivalent to the extension problem

$$\begin{array}{ccc} |\Lambda^n_i| & \xrightarrow{|f|} & X \\ \big\downarrow & \nearrow & \\ |\Delta^n| & & \end{array}$$

But this problem is solved by putting the dashed arrow to be $|f| \circ r$. □

Remark 2.44. Theorem 2.43 tells us that any topological space can be seen as a Kan complex. Analogously, any Kan complex gives rise to a topological space (which is simply given by taking the geometric realization). The adjunction $| - | \dashv \Pi_{\leqslant\infty}$ will then give rise to some sort of homotopical equivalence (Quillen equivalence). Details for this will be given in Chapter 5.

Homotopy groups (of any order) of a topological space are of course incredibly important in the study of homotopy theory for topological spaces. In the case of Kan complexes such homotopy groups also make sense. Indeed, we roughly sketch how such homotopy groups are defined: Let X be a Kan complex and pick a vertex $v \in X_0$. The 0-*th simplicial homotopy group* $\pi_0 X$ of X is defined to be the set of homotopy classes of vertices of X, i.e.,

$$\pi_0 X := \Big\{ [x] \mid x \in X_0 \Big\}$$

where $[x]$ denotes the set

$$\Big\{ x' \in X_0 \mid x' \sim x \Big\}$$

with $\sim$ being the smallest equivalence relation on X_0 such that $x \sim y$ if there exists $f \in X_1$ such that $d_1(f) = x$ and $d_0(f) = y$. In other words, $\pi_0 X$ is the coequalizer of

$$X_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} X_0$$

For $n \geqslant 1$, one defines $\pi_n(X, v)$ to be the set of homotopy classes of maps $\alpha\colon \Delta^n \to X$ relative boundary $\partial\Delta^n$. In other words, $\pi_n(X, v)$ is the set of equivalence classes $[\alpha]_{\text{rel}}$ where $\alpha\colon \Delta^n \to X$ is a simplicial map such that

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^0 \\ \big\downarrow & & \big\downarrow x \\ \Delta^n & \xrightarrow[\alpha]{} & X \end{array}$$

commutes, and where $[\alpha]_{\text{rel}} = [\beta]_{\text{rel}}$ if and only if there exists a homotopy $h\colon \Delta^1 \times \Delta^n \to X$ between α and β which respects the boundary condition:

$$\begin{array}{ccc} \Delta^1 \times \partial\Delta^n & \longrightarrow & \Delta^0 \\ \big\downarrow & & \big\downarrow x \\ \Delta^1 \times \Delta^n & \xrightarrow[h]{} & X \end{array}$$

Finally, one verifies that for $n \geqslant 1$ the sets $\pi_n(X, x)$ can be endowed with a group structure which is induced by the horn filling property of Kan complexes (see [26] or [15] for example). In fact, one can then also prove that we have a bijective correspondence of sets

$$\pi_0 X \cong \pi_0|X|$$

and, for all $n \geqslant 1$, group isomorphisms

$$\pi_n(X, v) \cong \pi_n(|X|, v)$$

i.e., the simplicial homotopy groups agree with the (topological) homotopy groups of the associated topological space obtained by applying geometric realization to X.

In analogy to topological spaces, we have the following definition:

Definition 2.45. Let $f\colon X \to Y$ be a simplicial map of Kan complexes. Then f is called a *simplicial weak equivalence* if f induces group isomorphisms

$$\pi_n(X, x) \xrightarrow{\quad f_\star \quad} \pi_n(Y, f(x))$$

for all $n \geqslant 1$ and vertices $x \in X_0$, and a bijection of sets

$$\pi_0(X, x) \xrightarrow{\quad f_\star \quad} \pi_0(Y, f(x))$$

Theorem 2.46 (Whitehead V1)**.** *Let $f\colon X \to Y$ be a simplicial map between Kan complexes. Then f is a simplicial weak equivalence if and only if f is a simplicial homotopy equivalence.*

Proof. See [28] Exercise 39.11. □

Theorem 2.47 (Whitehead V2)**.** *Let $f\colon X \to Y$ be simplicial map between Kan complexes. Then f is a simplicial homotopy equivalence if and only if all commutative squares*

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X \\ \big\downarrow & \overset{d}{\nearrow} & \big\downarrow f \\ \Delta^n & \longrightarrow & Y \end{array}$$

admit a lift $d\colon \Delta^n \to X$ such that the upper triangle commutes and the lower triangle commutes up to a homotopy relative boundary, i.e., there exists a homotopy $h\colon \Delta^1 \times \Delta^n \to Y$ from $f \circ d$ to the bottom map so that $h|_{\Delta^1 \times \partial\Delta^n}$ factors as

$$\begin{array}{ccc} \partial\Delta^n & & \\ \big\uparrow \pi_{\partial\Delta^n} & \searrow & \\ \Delta^1 \times \partial\Delta^n & \xrightarrow[h|_{\Delta^1 \times \partial\Delta^n}]{} & Y \end{array}$$

where $\pi_{\partial\Delta^n}$ denotes the corresponding projection.

Proof. See [28] Proposition 39.10. □

Weak equivalences can also be understood by means of a specific functor. In order to define this functor, we need some preliminary notions. We start off by defining the *last vertex map*: Recall, from Example 2.32, the functor Poset: $\Delta \to$ **Poset** which takes $[n]$ and maps it onto the poset of the ordinal $[n]$ denoted by Poset($[n]$). Also, recall that any ordinal $[n]$ can be interpreted as a category itself. The map max: Poset($[n]$) $\to [n]$ is given by

$$[v_0, \ldots, v_k] \mapsto v_k$$

where $[v_0, \ldots, v_k] \in \mathrm{Poset}([n])$ is an ordered tuple with $v_j \leqslant v_{j+1}$ for all j. The *last vertex map* $\lambda_{\Delta^n}\colon \mathrm{sd}\Delta^n \to \Delta^n$ is then defined as $\mathfrak{N}\max$. This map then gives rise to a simplicial map $X \to \mathrm{Ex}X$ for any simplicial set X (Ex was introduced in Example 2.32). Indeed, fix $X \in \mathrm{sSet}$. For any n-simplex $\sigma\colon \Delta^n \to X$, let $\rho_X(\sigma)$ denote the composite

$$\mathfrak{N}\mathrm{Poset}([n]) = \mathrm{sd}\Delta^n \xrightarrow{\lambda_{\Delta^n}} \mathfrak{N}[n] = \Delta^n \xrightarrow{\sigma} X$$

The map $\sigma \mapsto \rho_X(\sigma)$ then yields a natural transformation $X \to \mathrm{Ex}X$, where naturality follows from commutativity of

$$\begin{array}{ccc} \mathrm{Poset}([n]) & \xrightarrow{\max} & [n] \\ \downarrow{\scriptstyle f_\star} & & \downarrow{\scriptstyle f} \\ \mathrm{Poset}(\Delta^m) & \xrightarrow[\max]{} & [m] \end{array}$$

The simplicial maps ρ_X then assemble into a natural transformation $\mathrm{id}_{\mathrm{sSet}} \to \mathrm{Ex}$, i.e., the diagram

$$\begin{array}{ccc} \mathrm{dom}f & \xrightarrow{\rho_{\mathrm{dom}f}} & \mathrm{Ex}(\mathrm{dom}f) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Ex}f} \\ \mathrm{cod}f & \xrightarrow[\rho_{\mathrm{cod}f}]{} & \mathrm{Ex}(\mathrm{cod}f) \end{array}$$

commutes for all morphisms f in sSet.

Definition 2.48. The functor

$$\mathrm{Ex}^\infty\colon \mathrm{sSet} \longrightarrow \mathrm{sSet}$$

which sends $X \in \mathrm{sSet}$ to the colimit of the diagram

$$X \xrightarrow{\rho_X} \mathrm{Ex}X \xrightarrow{\rho_{\mathrm{Ex}X}} \mathrm{Ex}^2X \xrightarrow{\rho_{\mathrm{Ex}^2X}} \mathrm{Ex}^3 \longrightarrow \ldots$$

is referred to as *Kan's* Ex^∞*-functor*. The universal cocone that comes associated with the colimit $\mathrm{Ex}^\infty X$ then gives rise to a map $\rho_X^\infty\colon X \to \mathrm{Ex}^\infty X$. The construction $X \mapsto \rho_X^\infty$ determines a natural transformation $1_{\mathrm{sSet}} \to \mathrm{Ex}^\infty$.

The following theorem is of substantial importance to us:

Theorem 2.49. *A simplicial map f between Kan complexes is a simplicial weak equivalence if and only if $\mathrm{Ex}^\infty f$ is a simplicial homotopy equivalence.*

Proof. See [26] Corollary 3.3.6.8. □

This motivates the following definition:

Definition 2.50. A simplicial map f (not necessarily between Kan complexes) is called a *simplicial weak equivalence* if and only if $\mathrm{Ex}^\infty f$ is a simplicial homotopy equivalence.

Theorem 2.51. *The functor $\mathrm{Ex}^\infty\colon \mathrm{sSet} \to \mathrm{sSet}$ and the natural transformation $\rho^\infty\colon 1_{\mathrm{sSet}} \to \mathrm{Ex}^\infty$ enjoy the following properties:*

- *For every simplicial set $X \in \mathrm{sSet}$, the object $\mathrm{Ex}^\infty X$ is a Kan complex.*
- *For every simplicial set $X \in \mathrm{sSet}$, the map $\rho_X^\infty\colon X \to \mathrm{Ex}^\infty X$ is a weak homotopy equivalence.*
- *The functor Ex^∞ preserves weak equivalences, (trivial) fibrations, (trivial) cofibrations, and simplicial homotopy equivalences.*

- *For every Kan fibration of simplicial sets* $f \in \mathrm{sSet}$*, the induced morphism* $\mathrm{Ex}^{\infty} \colon \mathrm{Ex}^{\infty}(\mathrm{dom} f) \to \mathrm{Ex}^{\infty}(\mathrm{cod} f)$ *is a Kan fibration.*
- *The functor* $\mathrm{Ex}^{\infty} \colon \mathrm{sSet} \to \mathrm{sSet}$ *commutes with finite limits.*

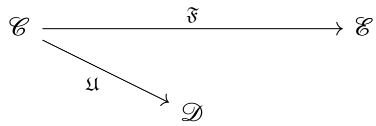
Proof. For details see [26] or [28]. □

3. Interlude on Kan Extensions

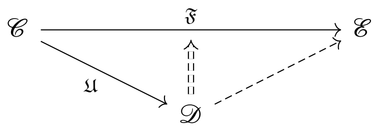
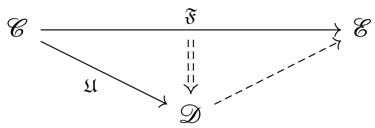
> The Road goes ever on and on Down from the door where it began. Now far ahead the Road has gone, And I must follow, if I can, Pursuing it with eager feet, Until it joins some larger way Where many paths and errands meet.
>
> J.R.R. Tolkien (The Fellowship of the Ring)

This chapter is based on the corresponding chapters in [23],[34] and [35].

3.1. **What even is a Kan Extension?** Let us assume we have a diagram of functors



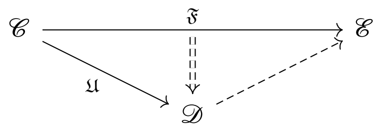
and let us view a category as the mathematical embodiment of a mathematical theory itself. A functor is then viewed as a translation of one mathematical theory to the language of another mathematical theory. Put differently, the functors $\mathfrak{F}$ and $\mathfrak{U}$ both model the mathematical theories $\mathscr{C}$ inside $\mathscr{E}$ and $\mathscr{D}$, respectively. Now the question is, if it is possible to model all of the theory $\mathscr{D}$ inside $\mathscr{E}$ by using the information of $\mathfrak{F}$ and $\mathfrak{U}$ in such a way so as to construct a functor that nicely blends in with all the data given? More concretely, we search for a functor $\mathscr{D} \to \mathscr{E}$ that should deserve to be called *extension of* $\mathfrak{F}$ *along* $\mathfrak{U}$. There are two canonical ways to define such a notion: The existence of such an extension functor $\mathfrak{E}\colon \mathscr{D} \to \mathscr{E}$ should either arrange for a comparison natural transformation $\mathfrak{E}\mathfrak{U} \to \mathfrak{F}$ or a comparison natural transformation $\mathfrak{F} \to \mathfrak{E}\mathfrak{U}$:

This motivates the following:

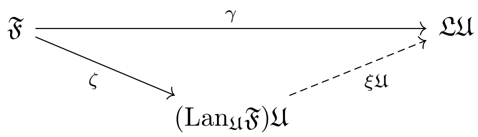
Definition 3.1. Let $\mathfrak{F}\colon \mathscr{C} \to \mathscr{E}, \mathfrak{U}\colon \mathscr{C} \to \mathscr{D}$ be functors between given categories.

- A *left Kan extension* of $\mathfrak{F}$ along $\mathfrak{U}$ is a functor $\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F}\colon \mathscr{D} \to \mathscr{E}$ together with a natural transformation $\zeta\colon \mathfrak{F} \to (\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F})\mathfrak{U}$ which collect into a universal pair $(\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F}, \zeta)$ for diagrams of the form

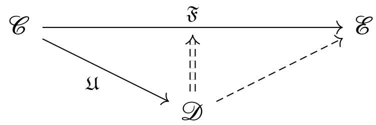
  Universality here means that for any other such pair
  $$(\mathfrak{L}\colon \mathscr{D} \to \mathscr{E}, \gamma\colon \mathfrak{F} \to \mathfrak{L}\mathfrak{U})$$

γ factors uniquely through ζ: There exists a unique $\xi\colon \mathrm{Lan}_{\mathfrak{U}}\mathfrak{F} \to \mathfrak{L}$ such that



commutes.

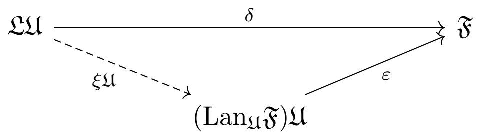
- A *right Kan extension* of $\mathfrak{F}$ along $\mathfrak{U}$ is a functor $\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}\colon \mathscr{D} \to \mathscr{E}$ together with a natural transformation $\varepsilon\colon (\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F})\mathfrak{U} \to \mathfrak{F}$ which collect into a universal pair $(\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}, \varepsilon)$ for diagrams of the form



Universality here means that for any other such pair

$$(\mathfrak{L}\colon \mathscr{D} \to \mathscr{E}, \delta\colon \mathfrak{L}\mathfrak{U} \to \mathfrak{F})$$

δ factors uniquely through ε: There exists a unique $\xi\colon \mathfrak{L} \to \mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}$ such that



commutes.

Passing to a higher set-theoretical universe SET, we can think of a left Kan extension of $\mathfrak{F}\colon \mathscr{C} \to \mathscr{E}$ along $\mathfrak{U}\colon \mathscr{C} \to \mathscr{D}$ as a representation for the functor

$$\mathscr{E}^{\mathscr{C}}(\mathfrak{F}, \mathfrak{U}^{\star})\colon \mathscr{E}^{\mathscr{D}} \to \mathrm{SET}, \qquad \mathfrak{L} \mapsto \mathscr{E}^{\mathscr{C}}(\mathfrak{F}, \mathfrak{L}\mathfrak{U})$$

where $\mathfrak{U}^{\star}\colon \mathscr{E}^{\mathscr{D}} \to \mathscr{E}^{\mathscr{C}}$ denotes the precomposition functor $- \circ \mathfrak{U}$. By the Yoneda Lemma any pair

$$\Big(\mathfrak{L}\colon \mathscr{D} \to \mathscr{E},\ \gamma \in \mathscr{E}^{\mathscr{C}}(\mathfrak{F}, \mathfrak{L}\mathfrak{U}) \cong \mathrm{Hom}(\mathscr{E}^{\mathscr{D}}(\mathfrak{L}, -), \mathscr{E}^{\mathscr{C}}(\mathfrak{F}, \mathfrak{U}^{\star})\Big)$$

as in the definition above, defines a natural transformation

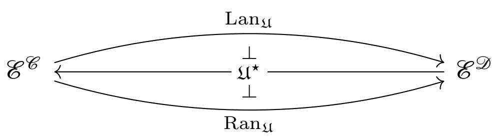
$$\mathscr{E}^{\mathscr{D}}(\mathfrak{L}, -) \xrightarrow{\gamma} \mathscr{E}^{\mathscr{C}}(\mathfrak{F}, \mathfrak{U}^{\star})$$

The universal property satisfied by the left Kan extension $(\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F}, \zeta)$ is then equivalent to the associated map

$$\mathscr{E}^{\mathscr{D}}(\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F}, -) \xrightarrow{\zeta} \mathscr{E}^{\mathscr{C}}(\mathfrak{F}, \mathfrak{U}^{\star})$$

being a natural isomorphism, i.e., $(\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F}, \zeta)$ represents the functor $\mathscr{E}^{\mathscr{C}}(\mathfrak{F}, \mathfrak{U}^{\star})$.

Proposition 3.2 ([35]). *If, for fixed $\mathfrak{U}\colon \mathscr{C} \to \mathscr{D}$ and $\mathscr{E}$, the left and right Kan extensions of any functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{E}$ along $\mathfrak{U}$ exist, then these define left and right adjoints to the pre-composition functor $\mathfrak{U}^{\star}\colon \mathscr{E}^{\mathscr{D}} \to \mathscr{E}^{\mathscr{C}}$: We have an adjoint correspondence*



and natural isomorphisms:

$$\mathscr{E}^{\mathscr{D}}(\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F}, \mathfrak{L}) \cong \mathscr{E}^{\mathscr{C}}(\mathfrak{F}, \mathfrak{L}\mathfrak{U}), \qquad \mathscr{E}^{\mathscr{C}}(\mathfrak{L}\mathfrak{U}, \mathfrak{F}) \cong \mathscr{E}^{\mathscr{D}}(\mathfrak{L}, \mathrm{Ran}_{\mathfrak{U}}\mathfrak{F})$$

3.2. **Pointwise and Absolute Kan Extensions.** The questions that should pop into one's mind right now are the following:

- Where do Kan extensions pop up in our quest to understand quantum field theory?
- How do we know certain Kan extensions will exist, and are there concrete formulas for these?

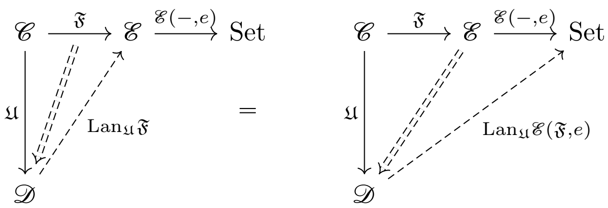
Since (higher) category theory is the language we use in order to describe quantum field theory, the first question is self-evident as essentially any notion in category theory may be seen to be a Kan extension. The second question also has an immediate answer (Proposition 3.5). First let us do some preparatory work and give some definitions:

Definition 3.3. Let

$$\mathscr{D} \xleftarrow{\mathfrak{U}} \mathscr{C} \xrightarrow{\mathfrak{F}} \mathscr{E}$$

be a pair of functors.

- A left Kan extension $\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F}$ along $\mathfrak{U}$ of $\mathfrak{F}$ is *pointwise* if it is preserved by all representable functors $\mathscr{E}(-,e)$ for all $e \in \mathscr{E}$:

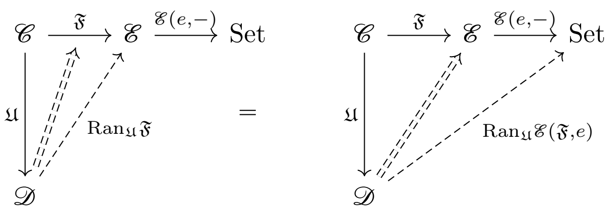


  In other words, if $(\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F},\ \zeta\colon \mathfrak{F} \to (\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F})\mathfrak{F})$ is a left Kan extension, then

$$(\mathscr{E}(\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F}, e),\ \mathscr{E}(\zeta, e))$$

  is the left Kan extension of $\mathscr{E}(\mathfrak{F}, e)$ along $\mathfrak{U}$.

- A right Kan extension $\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}$ along $\mathfrak{U}$ of $\mathfrak{F}$ is *pointwise* if it is preserved by all representable functors $\mathscr{E}(e,-)$ for all $e \in \mathscr{E}$:

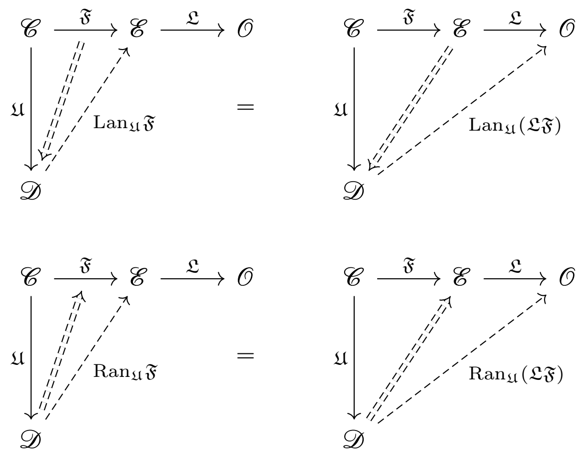


  In other words, if $(\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F},\ \zeta\colon (\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F})\mathfrak{F} \to \mathfrak{F})$ is a right Kan extension, then

$$(\mathscr{E}(e, \mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}),\ \mathscr{E}(e, \zeta))$$

  is the right Kan extension of $\mathscr{E}(e, \mathfrak{F})$ along $\mathfrak{U}$ for all $e \in \mathscr{E}$.

- A left/right Kan extension is said to be absolute if it is preserved by any functor $\mathfrak{L}\colon \mathscr{E} \to \mathscr{O}$ out of the codomain of $\mathfrak{F}$:



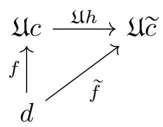
Following the notation of [35], the category $d \downarrow \mathfrak{U}$ for a functor $\mathfrak{U}\colon \mathscr{C} \to \mathscr{D}$ and $d \in \mathscr{D}$ is defined to be the category of elements of the functor $\mathscr{D}(d, \mathfrak{U})$, that is,

$$d \downarrow \mathfrak{U} := \mathbf{el}(\mathscr{D}(d, \mathfrak{U}))$$

We recall that objects in this category are given by pairs $(c \in \mathscr{C},\ f \in \mathscr{D}(d, \mathfrak{U}c))$ and a morphism

$$(c, f) \xrightarrow{h} (\widetilde{c}, \widetilde{f})$$

boils down to a morphism $h\colon c \to \widetilde{c}$ such that



commutes (see also Remark 2.11).

Lemma 3.4. *Given functors $\mathfrak{F}\colon \mathscr{C} \to \mathscr{E}$ and $\mathfrak{U}\colon \mathscr{C} \to \mathscr{D}$ with $\mathscr{D}$ and $\mathscr{E}$ locally small and an object $d \in \mathscr{D}$, there is a natural isomorphism*

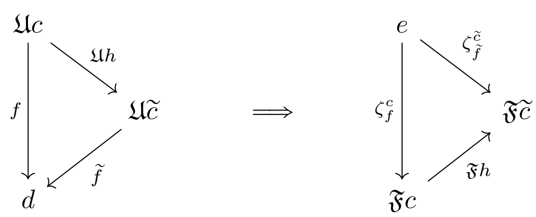
$$\mathrm{Cone}(e, \mathfrak{F}\Pi_{d\downarrow\mathfrak{U}}) \cong \mathrm{Set}^{\mathscr{C}}(\mathscr{D}(d, \mathfrak{U}), \mathscr{E}(e, \mathfrak{F}))$$

where $\Pi_{d\downarrow\mathfrak{U}}\colon d \downarrow \mathfrak{U} \to \mathscr{C}$ is the associated forgetful functor.

Proof. The set of cones $\mathrm{Cone}(e, \mathfrak{F}\Pi_{d\downarrow\mathfrak{U}})$ is equivalently given as the set of natural transformations from the constant diagram functor on e to $\mathfrak{F}\Pi_{d\downarrow\mathfrak{U}}$:

$$\mathrm{Cone}(e, \mathfrak{F}\Pi_{d\downarrow\mathfrak{U}}) = \mathscr{E}^{d\downarrow\mathfrak{U}}(\mathrm{const}(e), \mathfrak{F}\Pi_{d\downarrow\mathfrak{U}})$$

Any such cone is a family $(\zeta^c_f\colon \mathscr{D}(d, \mathfrak{U}c) \to \mathscr{E}(e, \mathfrak{F}c))_{(c,f)}$ such that



that is, if the LHS-triangle commutes then so does the RHS-triangle. However, this determines a natural family of functions $\zeta^c\colon \mathscr{D}(d,\mathfrak{U}c) \to \mathscr{E}(e,\mathfrak{F}c)$ so that for all $h\colon c \to \tilde{c}$ we have a commutative diagram

$$\begin{array}{ccc} \mathscr{D}(d,\mathfrak{U}c) & \xrightarrow{\zeta^c} & \mathscr{E}(e,\mathfrak{F}c) \\ {\scriptstyle(\mathfrak{U}h)_\star}\downarrow & & \downarrow{\scriptstyle(\mathfrak{F}h)_\star} \\ \mathscr{D}(d,\mathfrak{U}\tilde{c}) & \xrightarrow[\zeta^{\tilde{c}}]{} & \mathscr{E}(e,\mathfrak{F}\tilde{c}) \end{array}$$

□

Proposition 3.5. *Consider a pair of functors*

$$\mathscr{D} \xleftarrow{\mathfrak{U}} \mathscr{C} \xrightarrow{\mathfrak{F}} \mathscr{E}$$

such that $\mathscr{D}$ and $\mathscr{E}$ are locally small.

(*i*) *A right Kan extension of $\mathfrak{F}\colon \mathscr{C} \to \mathscr{E}$ along $\mathfrak{U}\colon \mathscr{C} \to \mathscr{D}$ is pointwise if and only if it can be computed by*

$$\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}(d) \cong \lim_{d\downarrow\mathfrak{U}} \mathfrak{F}\Pi_{d\downarrow\mathfrak{U}}$$

in which case, in particular, this limit exists.

(*ii*) *A left Kan extension of $\mathfrak{F}\colon \mathscr{C} \to \mathscr{E}$ along $\mathfrak{U}\colon \mathscr{C} \to \mathscr{D}$ is pointwise if and only if it can be computed by*

$$\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F}(d) \cong \operatorname*{colim}_{\mathfrak{U}\downarrow d} \mathfrak{F}\Pi_{\mathfrak{U}\downarrow d}$$

in which case, in particular, this colimit exists.

(*iii*) *If $\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}$ resp. $\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F}$ is pointwise and $\mathscr{E}$ is cotensored resp. tensored over* Set*, then we have natural isomorphisms (natural in $\mathfrak{F}$ and $\mathfrak{U}$):*

$$\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F} \cong \int^{c\in\mathscr{C}} よ(\mathfrak{U}c) \odot \mathfrak{F}c, \qquad \mathrm{Ran}_{\mathfrak{U}}\mathfrak{F} \cong \int_{c\in\mathscr{C}} よ(\mathfrak{U}c) \pitchfork \mathfrak{F}c$$

where $よ\colon \mathscr{D}^{\mathrm{op}} \to \mathrm{Set}^{\mathscr{D}}$ and $よ\colon \mathscr{D} \to \mathrm{Set}^{\mathscr{D}^{\mathrm{op}}}$ denote the contravariant and covariant Yoneda embedding, respectively.

Proof. If $\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}$ may be written by the above limit formula, then it is pointwise by preservation of limits of the hom-functor (in the covariant argument). Conversely, if $\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}$ is pointwise, then $\mathscr{E}(e,\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F})$ is the right Kan extension of $\mathscr{E}(e,\mathfrak{F})$ along $\mathfrak{U}$ for all $e \in \mathscr{E}$. The Yoneda Lemma combined with the defining universal property of the Kan extension yield

$$\begin{aligned} \mathscr{E}(e,\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}(d)) \cong \mathrm{Set}^{\mathscr{D}}(\mathscr{D}(d,-),\mathscr{E}(e,\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F})) &\cong \mathrm{Set}^{\mathscr{C}}(\mathscr{D}(d,\mathfrak{U}),\mathscr{E}(e,\mathfrak{F})) \\ &\cong \mathrm{Cone}(e,\mathfrak{F}\Pi_{d\downarrow\mathfrak{U}}) \end{aligned}$$

where the last isomorphism follows from Lemma 3.4. This proves the first statement. In order for

$$\int_{c\in\mathscr{C}} \mathscr{D}(d,\mathfrak{U}c) \pitchfork \mathfrak{F}c$$

to exist so that the end-formula for $\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}$ makes sense, we have to verify that the wedge functor induced by $\mathscr{D}(d,\mathfrak{U}) \pitchfork \mathfrak{F}$, defined in equation (2), is represented by $\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}(d)$. This, however, means that we only need to prove that the right Kan extension $\mathscr{E}(e,\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}(d))$ is given by

$$\Big\{\text{wedges } \star \dot{\to} \mathscr{D}(d,\mathfrak{U}) \pitchfork \mathscr{E}(e,\mathfrak{F})\Big\} \cong \mathrm{Set}\Big(\star, \int_{c\in\mathscr{C}} \mathscr{E}(e,\mathscr{D}(d,\mathfrak{U}c) \pitchfork \mathfrak{F}c)\Big)$$

$$
\begin{aligned}
&\cong \int_{c\in\mathscr{C}} \mathscr{E}(e, \mathscr{D}(d, \mathfrak{U}c) \pitchfork \mathfrak{F}c) \\
&\cong \int_{c\in\mathscr{C}} \mathrm{Set}(\mathscr{D}(d, \mathfrak{U}c), \mathscr{E}(e, \mathfrak{F}c)) \\
&\cong \mathrm{Set}^{\mathscr{C}}(\mathscr{D}(d, \mathfrak{U}), \mathscr{E}(e, \mathfrak{F}))
\end{aligned}
$$

The remaining claims follow by duality.

□

Remark 3.6. If we already know that the (co)end-formulas for left resp. right Kan extensions make sense, we may immediately calculate that the formulas above hold true: For example, if $\mathscr{E}$ is cocomplete, let $\mathfrak{L} \in \mathscr{E}^{\mathscr{D}}$ be any test functor. Then we have the following chain of natural isomorphisms (natural in $\mathfrak{F}$ and $\mathfrak{U}$):

$$
\begin{aligned}
\mathscr{E}^{\mathscr{D}}\Big(\int^{c\in\mathscr{C}} \text{よ}_{\mathscr{D}}(\mathfrak{U}c) \odot \mathfrak{F}c,\ \mathfrak{L}\Big) &\cong \int_{d\in\mathscr{D}} \mathscr{E}\Big(\int^{c\in\mathscr{C}} \mathscr{D}(\mathfrak{U}c, d) \odot \mathfrak{F}c,\ \mathfrak{L}d\Big) \\
&\cong \int_{d\in\mathscr{D}} \int_{c\in\mathscr{C}} \mathscr{E}(\mathscr{D}(\mathfrak{U}c, d) \odot \mathfrak{F}c,\ \mathfrak{L}d) \\
&\cong \int_{d\in\mathscr{D}} \int_{c\in\mathscr{C}} \mathrm{Set}(\mathscr{D}(\mathfrak{U}c, d),\ \mathscr{E}(\mathfrak{F}c, \mathfrak{L}d)) \\
&\cong \int_{c\in\mathscr{C}} \int_{d\in\mathscr{D}} \mathrm{Set}(\mathscr{D}(\mathfrak{U}c, d),\ \mathscr{E}(\mathfrak{F}c, \mathfrak{L}d)) \\
&\cong \int_{c\in\mathscr{C}} \mathrm{Set}^{\mathscr{D}}(\text{よ}_{\mathscr{D}}(\mathfrak{U}c), \mathscr{E}(\mathfrak{F}c, \mathfrak{L})) \\
&\cong \int_{c\in\mathscr{C}} \mathscr{E}(\mathfrak{F}c, (\mathfrak{L}\mathfrak{U})c) \\
&\cong \mathscr{E}^{\mathscr{C}}(\mathfrak{F}, \mathfrak{U}^{\star}\mathfrak{L})
\end{aligned}
$$

Example 3.7. Let us consider the unique functor

$$\mathscr{C} \xrightarrow{\quad !\quad} \star$$

where $\star$ denotes the terminal category. If the category $\mathscr{E}$ allows for the existence of the left adjoint $\mathrm{Lan}_! \dashv !^{\star}\colon \mathscr{E}^{\mathscr{C}} \to \mathscr{E}^{\star} \cong \mathscr{E}$ (e.g. $\mathscr{E}$ is cocomplete), then we get isomorphisms

$$\mathscr{E}^{\mathscr{C}}(\mathfrak{F}, \mathfrak{U}!) \cong \mathscr{E}^{\star}(\mathrm{Lan}_!\mathfrak{F}, \mathfrak{U})$$

where a functor $\mathfrak{U}\colon \star \to \mathscr{E}$ may be identified with just an object $e \in \mathscr{E}$, and the composition $\mathfrak{U}!$ is then nothing else than the constant diagram functor $\mathrm{const}(e)$ at e. Therefore, the left Kan extension $\mathrm{Lan}_!$ satisfies

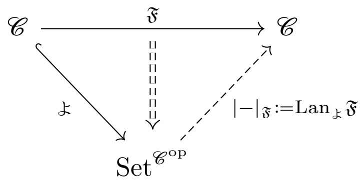
$$\mathscr{E}^{\mathscr{C}}(\mathfrak{F}, \mathrm{const}(e)) \cong \mathscr{E}(\mathrm{Lan}_!\mathfrak{F}, e)$$

which proves $\operatorname*{colim}_{\mathscr{C}} = \mathrm{Lan}_!$. Analogously, if $\mathscr{E}$ is complete, we obtain $\lim_{\mathscr{C}} = \mathrm{Ran}_!$. The full adjunction

$$\mathscr{C}^{\mathscr{D}} \; \begin{array}{c} \xrightarrow{\ \mathrm{colim}=\mathrm{Lan}_!\ } \\ \perp \\ \xleftarrow{\quad \mathrm{const} \quad} \\ \perp \\ \xrightarrow[\ \lim=\mathrm{Ran}_!\ ]{} \end{array} \; \mathscr{C}$$

will be quite important to us in later chapters.

Example 3.8. Recall Theorem 2.28: Let $\mathscr{D}$ be cocomplete. For a functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ we asked the question when there will exist a left adjoint for the nerve $\mathfrak{N}_{\mathfrak{F}}$ of $\mathfrak{F}$. The left Kan extension of $\mathfrak{F}$ along the Yoneda embedding $\text{よ}\colon \mathscr{C} \to \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$ will then give rise to the realization functor $|-|_{\mathfrak{F}}$:



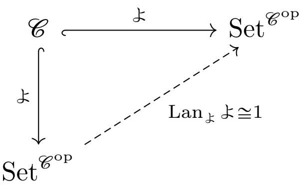
Indeed, by Proposition 3.5 we have

$$\mathrm{Lan}_{\text{よ}}\mathfrak{F}(\mathfrak{U}) = \int^{c\in\mathscr{C}} \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}(\text{よ}c, \mathfrak{U}) \odot \mathfrak{F}c \cong \int^{c\in\mathscr{C}} \mathfrak{U}c \odot \mathfrak{F}c$$

for any $\mathfrak{U} \in \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$. This exactly retrieves the formula from Theorem 2.28. Therefore,

$$\mathrm{Lan}_{\text{よ}}\mathfrak{F}\colon \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{C}\colon \mathfrak{N}_{\mathfrak{F}}$$

Corollary 3.9 (Density)**.** *For any small category $\mathscr{C}$, the identity functor defines the left Kan extension of the Yoneda embedding $\text{よ}\colon \mathscr{C} \to \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$ along itself:*



In particular,

$$\mathrm{Lan}_{\text{よ}}\text{よ}(\mathfrak{F}) \cong \mathrm{colim}\left(\mathbf{el}\mathfrak{F} \xrightarrow{\Pi} \mathscr{C} \xrightarrow{\text{よ}_{\mathscr{C}}} \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}\right) \cong \mathfrak{F}$$

Proof. We immediately see from the respective coend formula that $\mathrm{Lan}_{\text{よ}}\text{よ} \cong 1$:

$$\begin{aligned}\mathrm{Lan}_{\text{よ}}\text{よ}(\mathfrak{F}) &\cong \int^{c\in\mathscr{C}} \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}(\text{よ}c, \mathfrak{F}) \odot \text{よ}c \\ &\cong \int^{c\in\mathscr{C}} \mathfrak{F}c \odot \text{よ}c\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}(\mathrm{Lan}_{\text{よ}}\text{よ}(\mathfrak{F}), \mathfrak{U}) &\cong \int_c \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}(\mathfrak{F}c \odot \text{よ}c, \mathfrak{U}) \\ &\cong \int_c \mathrm{Set}(\mathfrak{F}c, \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}(\text{よ}c, \mathfrak{U})) \\ &\cong \int_c \mathrm{Set}(\mathfrak{F}c, \mathfrak{U}c) \\ &\cong \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}(\mathfrak{F}, \mathfrak{U})\end{aligned}$$

for all $\mathfrak{U} \in \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$, which proves $\mathrm{Lan}_{よ} よ(\mathfrak{F}) \cong \mathfrak{F}$. The remainder follows from $\mathbf{el}\mathfrak{F} \cong よ \downarrow \mathfrak{F}$ and the formula

$$\mathrm{colim}\left(よ \downarrow \mathfrak{F} \xrightarrow{\Pi} \mathscr{C} \xrightarrow{よ} \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}} \right) \cong \mathfrak{F}$$

which is implied by Proposition 3.5. ☐

The nerve realization Theorem 2.28 can also be understood in terms of Kan extensions, as we have seen in one of the previous examples. This begs the question whether or not *any* adjunction may be understood by means of Kan extensions. The answer is yes again:

Theorem 3.10. *Let $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ be a functor.*

- *The functor $\mathfrak{F}$ has a right adjoint if and only if $\mathrm{Lan}_{\mathfrak{F}} 1_{\mathscr{C}}$ exists and is preserved by $\mathfrak{F}$. In particular, $\mathrm{Lan}_{\mathfrak{F}} 1_{\mathscr{C}}$ is an absolute Kan extension.*
- *The functor $\mathfrak{F}$ has a left adjoint if and only if $\mathrm{Ran}_{\mathfrak{F}} 1_{\mathscr{C}}$ exists and is preserved by $\mathfrak{F}$. In particular, $\mathrm{Ran}_{\mathfrak{F}} 1_{\mathscr{C}}$ is an absolute Kan extension.*

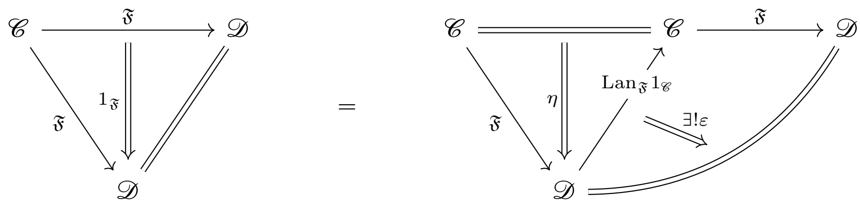
Proof. Suppose first that $\mathfrak{F}$ has a right adjoint $\mathfrak{U}\colon \mathscr{D} \to \mathscr{C}$. Then we obtain an adjunction

$$\mathscr{C}^{\mathscr{D}} \underset{\mathfrak{F}^{\star}}{\overset{\mathfrak{U}^{\star}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathscr{C}^{\mathscr{C}}$$

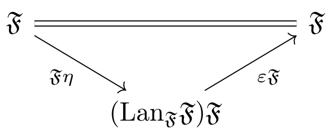
This follows since if $\eta\colon 1_{\mathscr{C}} \to \mathfrak{U}\mathfrak{F}$ and $\varepsilon\colon \mathfrak{F}\mathfrak{U} \to 1_{\mathscr{D}}$ are unit and counit, respectively, then $\eta^{\star}\colon 1_{\mathscr{C}^{\mathscr{C}}} \to \mathfrak{F}^{\star}\mathfrak{U}^{\star}$ and $\varepsilon^{\star}\colon \mathfrak{U}^{\star}\mathfrak{F}^{\star} \to 1_{\mathscr{C}^{\mathscr{D}}}$ give rise to adjunction unit and counit for $\mathfrak{U}^{\star} \dashv \mathfrak{F}^{\star}$. Now by uniqueness of adjoints, $\mathrm{Lan}_{\mathfrak{F}} \cong \mathfrak{U}^{\star}$. However, this implies that $\mathfrak{U}$ defines a left Kan extension of $1_{\mathscr{C}}$ along $\mathfrak{U}$. For any other functor $K\colon \mathscr{C} \to \mathscr{E}$ we have

$$\begin{aligned} \mathscr{E}^{\mathscr{C}}(K\mathrm{Lan}_{\mathfrak{F}}H, L) &\cong \mathscr{E}^{\mathscr{C}}(K_{\star}\mathfrak{U}^{\star}H, L) \\ &= \mathscr{E}^{\mathscr{C}}(\mathfrak{U}^{\star}(KH), L) \\ &\cong \mathscr{E}^{\mathscr{D}}(KH, \mathfrak{F}^{\star}L) \end{aligned}$$

for all $H\colon \mathscr{D} \to \mathscr{C}$ and all $L\colon \mathscr{C} \to \mathscr{E}$. This shows that $K\mathrm{Lan}_{\mathfrak{F}}H \cong \mathrm{Lan}_{\mathfrak{F}}(KH)$ and therefore $\mathrm{Lan}_{\mathfrak{F}}1_{\mathscr{C}}$ is absolute. Conversely, assume that the left Kan extension $(\mathrm{Lan}_{\mathfrak{F}}1_{\mathscr{C}}, \eta)$ exists and is preserved by $\mathfrak{F}$. Using the universal property of the Kan extension $\mathfrak{F}\mathrm{Lan}_{\mathfrak{F}}1_{\mathscr{C}} \cong \mathrm{Lan}_{\mathfrak{F}}\mathfrak{F}$ we obtain a unique factorization



In other words, we have a commutative diagram



proving one of the triangle identities. For the other identity, see [35]. ☐

4. Hom-objects and Enriched Category Theory

> My mind rebels at stagnation.
> Give me problems, give me work,
> give me the most abstruse
> cryptogram, or the most intricate
> analysis, and I am in my own
> proper atmosphere. But I abhor
> the dull routine of existence. I
> crave for mental exaltation.
>
> "The Sign of the Four" - Sir
> Arthur Conan Doyle

Enriched category theory is a powerful generalization of traditional category theory that allows us to work with categories enriched over other mathematical structures, such as sets, vector spaces, or topological spaces. In traditional category theory, a category comes with a set of morphisms for each pair of objects, but in enriched category theory, these Hom-sets are replaced with objects from a nice mathematical category (a *cosmos*). For example, a category enriched over topological spaces would have topological spaces of morphisms for each pair of objects. The notion of enriched category allows us to study the structure of categories in a more fine-grained way.

4.1. **Review on Symmetric Monoidal Categories.**

Definition 4.1. A *symmetric monoidal category* is a category $\mathscr{C}$ equipped with the following data:

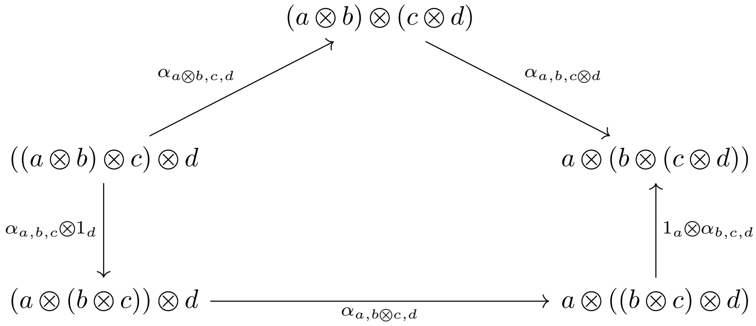
- A functor $\otimes\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, referred to as the *tensor product.*
- An object $\mathbb{1} \in \mathscr{C}$, called the *unit object.*
- Four natural isomorphisms

$$\alpha\colon\ \otimes \circ(\otimes \times 1_{\mathscr{C}}) \to \otimes \circ (1_{\mathscr{C}} \times \otimes)$$
$$\lambda\colon \mathbb{1} \otimes - \to 1_{\mathscr{C}}$$
$$\rho\colon\ - \otimes \mathbb{1} \to 1_{\mathscr{C}}$$
$$\beta\colon \otimes \to \otimes \circ \tau$$

  where $\tau\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C} \times \mathscr{C}$ is the *twist functor*, which takes a pair of objects (c, c') to (c', c). The above isomorphisms are referred to as *associator, left unitor, right unitor* and *braiding* in that order.

These are subject to coherence conditions, which demand that the following diagrams be commutative:

- *Pentagon axiom*:



- *Triangle identity*:

$$\begin{tikzcd}(a\otimes\mathbb{1})\otimes b \arrow[rr, "\alpha_{a,\mathbb{1},b}"] \arrow[dr, "\rho_a\otimes 1_b"'] & & a\otimes(\mathbb{1}\otimes b) \arrow[dl, "1_a\otimes\lambda_b"] \\ & a\otimes b & \end{tikzcd}$$

- *Hexagon identity*:

$$\begin{tikzcd}(a\otimes b)\otimes c \arrow[r, "\alpha_{a,b,c}"] \arrow[d, "\beta_{a,b}\otimes 1_c"'] & a\otimes(b\otimes c) \arrow[r, "\beta_{a,b\otimes c}"] & (b\otimes c)\otimes a \arrow[d, "\alpha_{b,c,a}"] \\ (b\otimes a)\otimes c \arrow[r, "\alpha_{b,a,c}"'] & b\otimes(a\otimes c) \arrow[r, "1_b\otimes\beta_{a,c}"'] & b\otimes(c\otimes a)\end{tikzcd}$$

- The *symmetry condition*:

$$\beta_{b,a}\beta_{a,b} = 1_{a\otimes b}$$

Example 4.2. (Symmetric) monoidal categories are abundant. We list some of the most popular examples:

- Let $\mathscr{C}$ be any category, then the category of endomorphisms $\mathrm{End}_{\mathscr{C}} := \mathscr{C}^{\mathscr{C}}$ on forms a monoidal category with the tensor product being composition of functors.
- The category Set is a symmetric monoidal category with the tensor product being the product functor $\times$.
- The category of $\mathbb{K}$-vector spaces $\mathrm{Vect}_{\mathbb{K}}$ yields a symmetric monoidal category with the tensor product being given by the usual tensor product of vector spaces.
- Consider the poset of non-negative real numbers $[0,\infty)\subset\mathbb{R}$. Viewing this poset as a category by means of $(a\geqslant b) \iff (\exists a\to b)$, we may define the associated tensor product to be addition of real numbers. This results in a symmetric monoidal category $([0,\infty),+,0)$ with tensor unit being 0.

Definition 4.3. a *symmetric monoidal functor* between symmetric monoidal categories $(\mathscr{C},\otimes,\mathbb{1},\lambda,\rho,\beta)$ and $(\mathscr{C}',\otimes',\mathbb{1}',\lambda',\rho',\beta')$ consists of

- a functor $\mathfrak{F}\colon\mathscr{C}\to\mathscr{C}'$,
- a natural isomorphism $\varphi\colon\ \otimes'\circ(\mathfrak{F}\times\mathfrak{F})\to\mathfrak{F}\circ\otimes$,
- an isomorphism $\varphi_1\colon\mathbb{1}'\to\mathfrak{F}\mathbb{1}$,

such that the diagrams

$$\begin{tikzcd}\mathfrak{F}(a\otimes b)\otimes'\mathfrak{F}c \arrow[r, "\varphi_{a\otimes b,c}"] & \mathfrak{F}((a\otimes b)\otimes c) \arrow[d, "\mathfrak{F}\alpha"] \\ (\mathfrak{F}a\otimes'\mathfrak{F}b)\otimes'\mathfrak{F}c \arrow[u, "\varphi_{a,b}\otimes' 1_{\mathfrak{F}c}"] \arrow[d, "\alpha"'] & \mathfrak{F}(a\otimes(b\otimes c)) \\ \mathfrak{F}a\otimes'(\mathfrak{F}b\otimes'\mathfrak{F}c) \arrow[r, "1_{\mathfrak{F}a}\otimes'\varphi_{b,c}"'] & \mathfrak{F}a\otimes'\mathfrak{F}(b\otimes c) \arrow[u, "\varphi_{a,b\otimes c}"']\end{tikzcd}$$

$$\begin{tikzcd}\mathbb{1}'\otimes'\mathfrak{F}a \arrow[r, "\lambda'_{\mathfrak{F}a}"] \arrow[d, "\varphi_1\otimes 1_{\mathfrak{F}a}"'] & \mathfrak{F}a \arrow[d, "\mathfrak{F}\lambda_a"] \\ \mathfrak{F}\mathbb{1}\otimes'\mathfrak{F}a \arrow[r, "\varphi_{1,a}"'] & \mathfrak{F}(\mathbb{1}\otimes a)\end{tikzcd} \qquad \begin{tikzcd}\mathfrak{F}a\otimes'\mathbb{1}' \arrow[r, "\rho'_{\mathfrak{F}a}"] \arrow[d, "1_{\mathfrak{F}a}\otimes\varphi_1"'] & \mathfrak{F}a \arrow[d, "\mathfrak{F}\rho_a"] \\ \mathfrak{F}a\otimes'\mathfrak{F}\mathbb{1} \arrow[r, "\varphi_{a,1}"'] & \mathfrak{F}(a\otimes\mathbb{1})\end{tikzcd}$$

$$
\begin{array}{ccc}
\mathfrak{F}a \otimes' \mathfrak{F}b & \xrightarrow{\beta_{\mathfrak{F}a,\mathfrak{F}b}} & \mathfrak{F}b \otimes' \mathfrak{F}a \\
\downarrow{\scriptstyle \varphi_{a,b}} & & \downarrow{\scriptstyle \varphi_{b,a}} \\
\mathfrak{F}(a \otimes b) & \xrightarrow[\mathfrak{F}\beta_{a,b}]{} & \mathfrak{F}(b \otimes a)
\end{array}
$$

commute for all $a, b, c \in \mathscr{C}$.

Definition 4.4. Let $(\mathfrak{F}, \varphi, \varphi_1)$ and $(\mathfrak{U}, \psi, \psi_1)$ be symmetric monoidal functors $\mathscr{C} \to \mathscr{D}$. A natural transformation $\zeta\colon \mathfrak{F} \to \mathfrak{U}$ is called a *symmetric monoidal transformation* if the diagrams

$$
\begin{array}{ccc}
\mathfrak{F}a \otimes_{\mathscr{D}} \mathfrak{F}b & \xrightarrow{\zeta_a \otimes_{\mathscr{D}} \zeta_b} & \mathfrak{U}a \otimes_{\mathscr{D}} \mathfrak{U}b \\
\downarrow{\scriptstyle \varphi_{a,b}} & & \downarrow{\scriptstyle \psi_{a,b}} \\
\mathfrak{F}(a \otimes_{\mathscr{C}} b) & \xrightarrow[\zeta_{a \otimes_{\mathscr{C}} b}]{} & \mathfrak{U}(a \otimes_{\mathscr{C}} b)
\end{array}
\qquad
\begin{array}{ccc}
& \mathbb{1}_{\mathscr{D}} & \\
{\scriptstyle \varphi_1}\swarrow & & \searrow{\scriptstyle \psi_1} \\
\mathfrak{F}\mathbb{1}_{\mathscr{C}} & \xrightarrow[\zeta_{\mathbb{1}_{\mathscr{C}}}]{} & \mathfrak{U}\mathbb{1}_{\mathscr{C}}
\end{array}
$$

commute for all $a, b \in \mathscr{C}$.

4.1.1. *Duals.*

Definition 4.5. Let $\mathscr{C}$ be a symmetric monoidal category.

- An object $c \in \mathscr{C}$ is said to have a *dual* if there exists an object $c^\dagger \in \mathscr{C}$ such that we have an adjunction

$$
\mathscr{C} \underset{-\otimes c^\dagger}{\overset{-\otimes c}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathscr{C}
$$

- $\mathscr{C}$ is said to have *duals*, if every object $c \in \mathscr{C}$ has a dual.

We realize that an adjunction $- \otimes c \dashv - \otimes c^\dagger$ as above induces corresponding unit and counit maps

$$
\eta\colon 1_{\mathscr{C}} \to c^\dagger \otimes c \otimes -, \qquad \varepsilon\colon c^\dagger \otimes c \otimes - \to 1_{\mathscr{C}}
$$

However, these natural transformations are already fully determined by the respective components $\eta_{\mathbb{1}}$ and $\varepsilon_{\mathbb{1}}$. Indeed, η and ε, being the corresponding unit and counit of an adjunction, satisfy the triangle axioms:

$$
\begin{array}{ccc}
c \otimes - & \xrightarrow{c \otimes \eta} & c \otimes c^\dagger \otimes c \otimes - \\
\| & \swarrow{\scriptstyle \varepsilon \otimes c} & \\
c \otimes - & &
\end{array}
\qquad
\begin{array}{ccc}
c^\dagger \otimes - & \xrightarrow{c^\star \otimes \eta} & c^\dagger \otimes c \otimes c^\dagger \otimes - \\
\| & \swarrow{\scriptstyle \varepsilon \otimes c^\dagger} & \\
c^\dagger \otimes - & &
\end{array}
$$

In particular, we obtain commutative diagrams

$$
\begin{array}{ccc}
c & \xrightarrow{c \otimes \eta_{\mathbb{1}}} & c \otimes c^\dagger \otimes c \\
\| & \swarrow{\scriptstyle \varepsilon_{\mathbb{1}} \otimes c} & \\
c & &
\end{array}
\qquad
\begin{array}{ccc}
c^\dagger & \xrightarrow{c^\dagger \otimes \eta_{\mathbb{1}}} & c^\dagger \otimes c \otimes c^\dagger \\
\| & \swarrow{\scriptstyle \varepsilon_{\mathbb{1}} \otimes c^\dagger} & \\
c^\dagger & &
\end{array}
$$

where we have made extensive use of the natural isomorphism $- \otimes \mathbb{1} \cong 1_{\mathscr{C}}$. From such commutative diagrams for $\eta_{\mathbb{1}}$ and $\varepsilon_{\mathbb{1}}$ as above, we may recover η and ε by defining

$$\eta_{c'} := \eta_{\mathbb{1}} \otimes 1_{c'}, \qquad \varepsilon_{c'} := \varepsilon_{\mathbb{1}} \otimes 1_{c'}$$

Example 4.6. Consider the symmetric monoidal category of $\mathbb{K}$-vector spaces $\mathrm{Vect}_{\mathbb{K}}$. It may be shown that a vector space $V \in \mathrm{Vect}_{\mathbb{K}}$ is dualizable if and only if V is finite dimensional. Let us show one direction of this equivalence. Suppose V is finite dimensional and set $V^\dagger := \mathrm{Vect}_{\mathbb{K}}(V, \mathbb{K})$. Recall that the tensor unit of $\mathrm{Vect}_{\mathbb{K}}$ is $\mathbb{K}$ itself and therefore it suffices to construct appropriate linear maps

$$\eta_{\mathbb{K}} \colon \mathbb{K} \to V^\dagger \otimes V, \qquad \varepsilon_{\mathbb{K}} \colon V^\dagger \otimes V \to \mathbb{K}$$

The definition for $\varepsilon_{\mathbb{K}}$ is canonical. For $\psi \in V^\dagger$ and $v \in V$, we set

$$\varepsilon_{\mathbb{K}}(\psi \otimes v) := \psi(v)$$

and extend this map linearly to all of $V^\dagger \otimes V$. For the definition of η, we fix a (finite) basis $\{v_i\}_{i=1}^d$ for V and define

$$\eta_{\mathbb{K}}(1) := \sum_i v_i^\dagger \otimes v$$

where the $\{v_i^\dagger\}$ denotes the corresponding dual basis. It may be shown that the definition of $\eta_{\mathbb{K}}$ is independent of the specific choice of a basis. Moreover, it is not hard to see that the two maps thus defined satisfy the triangle identities, which shows that the full symmetric monoidal sub-category of finite dimensonsal vector spaces $\mathrm{vect}_{\mathbb{K}}$ has all duals.

Example 4.7. Consider the symmetric monoidal category Set which has the product as its tensor product. Note that except for the singleton $\star \in \mathrm{Set}$, no other object $S \in \mathrm{Set}$ (which has cardinality greater than 1) has a dual. Indeed, suppose $S \in \mathrm{Set}$ with $|S| > 1$, has a dual $S^\dagger$, then

$$\mathrm{Set}(A \times S, B) \cong \mathrm{Set}(A, B \times S^\dagger)$$

In particular, for $A = \star$ a singleton, we would obtain

$$\mathrm{Set}(S, B) \cong \mathrm{Set}(\star, B \times S^\dagger) \cong B \times S^\dagger$$

for all $B \in \mathrm{Set}$, which is impossible unless $S = S^\dagger = \star$.

4.2. **Internal Homs.** This chapter is based on [38] and the corresponding Nlab-article on internal homs.

As a motivating example let us, just briefly so, consider the category Set. For $X, Y, Z \in \mathrm{Set}$ we readily have the canonical natural isomorphism

$$\mathrm{Set}(X \times Y, Z) \xrightarrow{\cong} \mathrm{Set}(X, \mathrm{Set}(Y, Z))$$

which maps a function $f \colon X \times Y \to Z$ to the induced function $\widetilde{f} \colon X \to \mathrm{Set}(Y, Z)$ which is given by $\widetilde{f}(x) := f(x, -)$. We note then that the category of sets is special in that for each $Y \in \mathrm{Set}$ the functor $- \times Y$ has a right adjoint $\mathrm{Set}(Y, -)$. To spell out a triviality concretely: One of the most unique properties of Set is that for any $X, Y \in \mathrm{Set}$ the Hom-set $\mathrm{Set}(X, Y)$ is again an object in Set. This is something we would also like to have in an arbitrary category $\mathscr{C}$, i.e., we aspire to get, for each pair $X, Y \in \mathscr{C}$, a hom-object $[X, Y] \in \mathscr{C}$ (rather than just a set) which should, in some sense, contain the same information as the usual Hom-set $\mathscr{C}(X, Y)$ with the distinction of being even richer in that it is also an object in the category itself.

Definition 4.8. Let $\mathscr{C}$ be a symmetric monoidal category. An *internal hom* in $\mathscr{C}$ is a functor

$$[-,-]\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$$

such that for every object $c \in \mathscr{C}$ we have a pair of adjoint functors

$$\mathscr{C} \underset{[c,-]}{\overset{c\otimes -}{\rightleftarrows}} \mathscr{C} \qquad (\bot)$$

If an internal hom exists in $\mathscr{C}$, we call $\mathscr{C}$ a *closed symmetric monoidal category.*

Remark 4.9. We note that the concept of an internal hom generalizes the notion of a dual.

Proposition 4.10. *In a closed symmetric monoidal category $\mathscr{C}$ there are natural isomorphisms*

$$[a,[b,c]] \cong [a \otimes b, c]$$

Proof. Let $x \in \mathscr{C}$ be any object. We have the following chain of natural isomorphisms

$$\begin{aligned} \mathscr{C}(x,[a\otimes b,c]) &\cong \mathscr{C}(x\otimes(a\otimes b),c) \cong \mathscr{C}((x\otimes a)\otimes b,c) \\ &\cong \mathscr{C}(x\otimes a,[b,c]) \cong \mathscr{C}(x,[a,[b,c]]) \end{aligned}$$

Since x was arbitrary the claim follows from fully faithfulness of the Yoneda embedding. $\square$

Proposition 4.11. *Let $\mathscr{C}$ be a closed symmetric monoidal category with internal hom-bifunctor $[-,-]$. Then this bifunctor preserves limits in the second variable, and sends colimits in the first variable to limits:*

$$[c, \lim_{\mathscr{J}} \mathfrak{F}j] \cong \lim_{\mathscr{J}}[c,\mathfrak{F}j], \qquad [\operatorname{colim}_{\mathscr{J}} \mathfrak{F}j, c] \cong \lim_{\mathscr{J}}[\mathfrak{F}j,c]$$

for any (small) functor $\mathfrak{F}\colon \mathscr{J} \to \mathscr{C}$ and any object $c \in \mathscr{C}$.

Proof. Since $[c,-]$ is a right adjoint we immediately obtain preservation of limits in the covariant slot of $[-,-]$. For the other case, let $\mathfrak{F}\colon \mathscr{J} \to \mathscr{C}$ be a small diagram, and let $a \in \mathscr{C}$ be fixed. Then we have the following chain of natural isomorphisms

$$\mathscr{C}(a,[\operatorname{colim}_{\mathscr{J}} \mathfrak{F}j,c]) \cong \mathscr{C}(a\otimes \operatorname{colim}_{\mathscr{J}} \mathfrak{F}j,c) \cong \mathscr{C}(\operatorname{colim}_{\mathscr{J}}(a\otimes \mathfrak{F}j),c) \cong \lim_{\mathscr{J}} \mathscr{C}(a\otimes \mathfrak{F}j,c)$$

where we also made use of the fact that $a \otimes -$ is a left adjoint and thus preserves colimits. $\square$

Closed symmetric monoidal categories are not all that rare. The main examples we will concern ourselves with are those induced by cartesian closed categories:

Definition 4.12. A category $\mathscr{C}$ is *cartesian closed* if

- it has finite products (this also implies the existence of a terminal object).
- for each $c \in \mathscr{C}$, the product-functor $c \times -\colon \mathscr{C} \to \mathscr{C}$ admits a right adjoint $[c,-]\colon \mathscr{C} \to \mathscr{C}$:

  $$\mathscr{C} \underset{[c,-]}{\overset{c\times -}{\rightleftarrows}} \mathscr{C} \qquad (\bot)$$

  For $c, c' \in \mathscr{C}$ the resulting object $[c,c']$ will be referred to as the *internal hom* (or *exponential*) from c to c'.

Remark 4.13. Any cartesian closed category $\mathscr{C}$ induces a closed symmetric monoidal category: The tensor product is simply defined to be the product bifunctor $- \times -$. The unit object is given by the terminal object $\star \in \mathscr{C}$. Associators, left and right unitors and the braiding are induced by the obvious natural isomorphisms.

Example 4.14. Let us list some examples of cartesian closed categories (and thereby also of closed symmetric monoidal categories):

- Set is cartesian closed with internal hom $[X, Y] := \mathrm{Set}(X, Y)$.
- The category of small categories Cat is cartesian closed: For $\mathscr{A}, \mathscr{B} \in \mathrm{Cat}$ the internal hom $[\mathscr{A}, \mathscr{B}] := \mathscr{B}^{\mathscr{A}}$ is simply defined to be the corresponding functor category. The associated natural isomorphisms read
$$\mathrm{Cat}(\mathscr{A}, \mathscr{C}^{\mathscr{B}}) \cong \mathrm{Cat}(\mathscr{A} \times \mathscr{B}, \mathscr{C}) \cong \mathrm{Cat}(\mathscr{B}, \mathscr{C}^{\mathscr{A}})$$
- For any small category $\mathscr{C}$, the category $\widehat{\mathscr{C}} := \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$ is cartesian closed. Indeed, for $\mathfrak{F}, \mathfrak{U} \in \widehat{\mathscr{C}}$, the value of $[\mathfrak{F}, \mathfrak{U}]$ at $c \in \mathscr{C}$ must be defined by
$$[\mathfrak{F}, \mathfrak{U}](c) \cong \widehat{\mathscr{C}}(よ_{\mathscr{C}} c, [\mathfrak{F}, \mathfrak{U}]) \cong \widehat{\mathscr{C}}(\mathfrak{F} \times よ_{\mathscr{C}} c, \mathfrak{U})$$
- In particular, the category $\mathrm{sSet} := \mathrm{Set}^{\Delta^{\mathrm{op}}}$ is cartesian closed with internal hom
$$\mathrm{sSet} \ni [X, Y] := \mathrm{sSet}(X \times よ_{\Delta}, Y)$$
for $X, Y \in \mathrm{sSet}$.

Definition 4.15. For $\mathscr{C}$ a closed symmetric monoidal category, the *underlying set functor* is the functor
$$(-)_0 := \mathscr{C}(\mathbb{1}, -) \colon \mathscr{C} \longrightarrow \mathrm{Set}$$
represented by the unit object $\mathbb{1} \in \mathscr{C}$.

Remark 4.16. Since $(-)_0$ is given as a covariant representable functor, this functor preserves limits.

Lemma 4.17. *For any pair of objects $c, c' \in \mathscr{C}$ in a closed symmetric monoidal category, the underlying set of the internal hom $[c, c']$ is $\mathscr{C}(c, c')$, i.e.:*
$$[c, c']_0 \cong \mathscr{C}(c, c')$$

Proof. By definition
$$[c, c']_0 = \mathscr{C}(\mathbb{1}, [c, c']) \cong \mathscr{C}(\mathbb{1} \otimes c, c') \cong \mathscr{C}(c, c')$$
where the last isomorphism follows from $\mathbb{1} \otimes c \cong c$. □

4.3. **Enriched Category Theory.** For the following section we will follow the exposition given in the appendix of [38].

Throughout, we shall fix a complete and cocomplete closed symmetric monoidal category $(\mathscr{V}, \otimes, \mathbb{1})$ to serve as the base for enrichment.

Definition 4.18. A $\mathscr{V}$*-enriched category* or $\mathscr{V}$*-category* $\mathscr{C}$ is given by

- a collection of objects
- for each pair of objects $x, y \in \mathscr{C}$ an hom-object $\mathscr{C}(x, y) \in \mathscr{V}$
- for each $x \in \mathscr{C}$ a specified identity element encoded by a map $1_x \colon \mathbb{1} \to \mathscr{C}(x, x)$, and for each $x, y, z \in \mathscr{C}$ a specified composition map $\circ \colon \mathscr{C}(y, z) \otimes$

$\mathscr{C}(x,y) \to \mathscr{C}(x,z) \in \mathscr{V}$ satisfying the associativity and unit conditions which demand that the following two squares should commute:

$$\begin{array}{ccc}
\mathscr{C}(y,z)\otimes\mathscr{C}(x,y)\otimes\mathscr{C}(w,x) & \xrightarrow{\circ\otimes\mathrm{id}} & \mathscr{C}(x,z)\otimes\mathscr{C}(w,x) \\
{\scriptstyle \mathrm{id}\otimes\circ}\downarrow & & \downarrow{\scriptstyle\circ} \\
\mathscr{C}(y,z)\otimes\mathscr{C}(w,y) & \xrightarrow[\circ]{} & \mathscr{C}(w,z)
\end{array}$$

$$\begin{array}{ccc}
\mathscr{C}(x,y) & \xrightarrow{\mathrm{id}\otimes 1_x} & \mathscr{C}(x,y)\otimes\mathscr{C}(x,x) \\
{\scriptstyle 1_y\otimes\mathrm{id}}\downarrow & \searrow\!\!= & \downarrow{\scriptstyle\circ} \\
\mathscr{C}(y,y)\otimes\mathscr{C}(x,y) & \xrightarrow[\circ]{} & \mathscr{C}(x,y)
\end{array}$$

Remark 4.19. It is immediate from the definition that a locally small 1-category defines a category enriched in Set.

Example 4.20. If $\mathscr{C}$ and $\mathscr{D}$ are $\mathscr{V}$-enriched categories, then the corresponding $\mathscr{V}$-enriched product category $\mathscr{C}\times\mathscr{D}$ has as its set objects pairs (c,d) for $c\in\mathscr{C}$ and $d\in\mathscr{D}$. For (c,d) and (c',d') two objects as above, the corresponding hom-object is given by

$$\mathscr{C}(c,c')\otimes\mathscr{D}(d,d')$$

The composition operation is the braiding followed by the tensor product of the respective composition operations:

$$\begin{array}{c}
(\mathscr{C}\times\mathscr{D})((c_1,d_1),(c_2,d_2))\otimes(\mathscr{C}\times\mathscr{D})((c_2,d_2),(c_3,d_3)) \\
\| \\
(\mathscr{C}(c_1,c_2)\otimes\mathscr{D}(d_1,d_2))\otimes(\mathscr{C}(c_2,c_3)\otimes\mathscr{D}(d_2,d_3)) \\
\downarrow{\scriptstyle\cong} \\
(\mathscr{C}(c_1,c_2)\otimes\mathscr{C}(c_2,c_3))\otimes(\mathscr{D}(d_1,d_2)\otimes\mathscr{D}(d_2,d_3)) \\
\downarrow{\scriptstyle\circ_{\mathscr{C}}\otimes\circ_{\mathscr{D}}} \\
\mathscr{C}(c_1,c_3)\otimes\mathscr{D}(d_1,d_3) = (\mathscr{C}\times\mathscr{D})((c_1,d_1),(c_3,d_3))
\end{array}$$

Example 4.21. Let $\mathscr{C}$ be a $\mathscr{V}$-enriched category. The *opposite $\mathscr{V}$-enriched category* $\mathscr{C}^{\mathrm{op}}$ has the same objects as $\mathscr{C}$, with hom-objects $\mathscr{C}^{\mathrm{op}}(c,c') := \mathscr{C}(c',c)$ and with composition given by braiding followed by composition in $\mathscr{C}$:

$$\begin{array}{c} \mathscr{C}^{\mathrm{op}}(c_1, c_2) \otimes \mathscr{C}^{\mathrm{op}}(c_2, c_3) \\ \Big\| \\ \mathscr{C}(c_2, c_1) \otimes \mathscr{C}(c_3, c_2) \\ \Big\downarrow \cong \\ \mathscr{C}(c_3, c_2) \otimes \mathscr{C}(c_2, c_1) \\ \Big\downarrow \circ_{\mathscr{C}} \\ \mathscr{C}(c_3, c_1) = \mathscr{C}^{\mathrm{op}}(c_1, c_3) \end{array}$$

Example 4.22. View $\mathbb{R}_{\geqslant 0}$ as a symmetric monoidal category where the monoidal product is given by addition of non-negative real numbers (recall $x \geqslant y \iff \exists x \to y$). We note that $\mathbb{R}_{\geqslant 0}$ is closed, since for $a \geqslant b \in \mathbb{R}_{\geqslant 0}$ the corresponding internal hom may be given by

$$[a, b] := a - b \in \mathbb{R}_{\geqslant 0}$$

Hence it makes sense to talk of $\mathbb{R}_{\geqslant 0}$-enriched categories. Suppose X is a $\mathbb{R}_{\geqslant 0}$-enriched category. Then X consists of a set of objects X_0 and for each $x, y \in X_0$ we get a hom-object $X(x, y) \in \mathbb{R}_{\geqslant 0}$. From the defining conditions of what it means to be $\mathbb{R}_{\geqslant 0}$-enriched, we obtain the triangle inequality:

$$X(x, z) + X(w, x) \geqslant X(w, z)$$

In other words, $X(-, -)$ is reminiscient to a metric. In fact, a $\mathbb{R}_{\geqslant 0}$-enriched category is nothing more than a *Lawvere metric space* (only the symmetry condition is missing from a typical metric space). An $\mathbb{R}_{\geqslant 0}$-enriched functor (see Definition 4.26) $f \colon X \to Y$ between two Lawvere metric spaces is nothing more than a map of sets $X_0 \to Y_0$ such that

$$X(x, y) \geqslant Y(fx, fy)$$

This could be considered as a continuous map with respect to the corresponding induced topologies.

Example 4.23. The category Cat is cartesian closed, hence in particular a symmetric monoidal category. A Cat-enriched category is called a *strict* 2*-category* and the corresponding category of Cat-enriched categories, denoted St-2-Cat, is the category of strict 2-categories. One notes that St-2-Cat is again cartesian closed, and hence it makes sense to talk about St-2-Cat-enriched categories, which in turn yields the notion of a *strict* 3*-category*. More generally, a *strict* n*-category* is a St-$(n-1)$-Cat-enriched category, where St-$(n-1)$-Cat is the category of St-$(n-2)$-Cat-enriched categories. We note that the cartesian structure on the category St-$(n-1)$Cat is just taking products of strict $(n-1)$-categories, while cartesian closedness follows from Proposition 4.53 (the corresponding internal Hom is given by the *Day internal Hom*).

Any $\mathscr{V}$-category $\mathscr{C}$ has an underlying category:

Definition 4.24. If $\mathscr{C}$ is a $\mathscr{V}$-category, its *underlying category* $\mathscr{C}_0$ is the 1-category with the same collection of objects and with Hom-sets defined by applying the underlying set functor $(-)_0 \colon \mathscr{V} \to \mathrm{Set}$ to the hom-objects $\mathscr{C}(x,y) \in \mathscr{V}$. The identity arrow $1_x \colon \mathbb{1} \to \mathscr{C}(x,x)$ is already an element of $\mathscr{C}(x,x)_0 := \mathscr{V}(\mathbb{1}, \mathscr{C}(x,x))$ and the composite of two arrows $f \colon 1 \to \mathscr{C}(x,y)$ and $g \colon 1 \to \mathscr{C}(y,z)$ is defined to be the composition

$$1 \xrightarrow{\quad g \otimes f \quad} \mathscr{C}(y,z) \times \mathscr{C}(x,y) \xrightarrow{\quad \circ \quad} \mathscr{C}(x,z)$$

Proposition 4.25. *A cartesian closed category $\mathscr{V}$ defines a $\mathscr{V}$-category with*

- *the same objects as $\mathscr{V}$.*
- *hom object in $\mathscr{V}$ from a to b being the internal hom $[a,b] \in \mathscr{V}$.*
- *the identity map $1_a \colon \star \to [a,a]$ and composition map $\circ \colon [b,c] \times [a,b] \to [a,c]$ defined by taking the transposes of*

$$\star \times a \xrightarrow{\cong} a, \qquad [b,c] \times [a,b] \times a \xrightarrow{\mathrm{id} \times \mathrm{ev}} [b,c] \times b \xrightarrow{\mathrm{ev}} c$$

where the evaluation map $\mathrm{ev}_{a,b} \colon [a,b] \times a \to b$ is the $\Big(- \times a \dashv [a,-] \Big)$-adjunct of the identity $1_{[a,b]} \colon [a,b] \to [a,b]$.

Proof. See [38] page 398 Lemma A.2.3. □

Definition 4.26. A *$\mathscr{V}$-enriched functor* or *$\mathscr{V}$-functor* $\mathfrak{F} \colon \mathscr{C} \to \mathscr{D}$ is given by

- a mapping on objects that carries each $x \in \mathscr{C}$ to an object $\mathfrak{F}x \in \mathscr{D}$
- for each pair of objects $x, y \in \mathscr{C}$, a morphism $\mathfrak{F}_{x,y} \colon \mathscr{C}(x,y) \to \mathscr{D}(\mathfrak{F}x, \mathfrak{F}y) \in \mathscr{V}$ so that the $\mathscr{V}$-functoriality diagrams commute:

$$\begin{array}{ccc} \mathscr{C}(y,z) \otimes \mathscr{C}(x,y) & \xrightarrow{\quad \circ \quad} & \mathscr{C}(x,z) \\ \downarrow{\scriptstyle \mathfrak{F}_{y,z} \otimes \mathfrak{F}_{x,y}} & & \downarrow{\scriptstyle \mathfrak{F}_{x,z}} \\ \mathscr{D}(\mathfrak{F}y, \mathfrak{F}z) \otimes \mathscr{D}(\mathfrak{F}x, \mathfrak{F}y) & \xrightarrow[\quad \circ \quad]{} & \mathscr{D}(\mathfrak{F}x, \mathfrak{F}z) \end{array} \qquad \begin{array}{ccc} \mathbb{1} & \xrightarrow{\quad 1_x \quad} & \mathscr{C}(x,x) \\ & \searrow^{1_{\mathfrak{F}x}} & \downarrow{\scriptstyle \mathfrak{F}_{x,x}} \\ & & \mathscr{D}(\mathfrak{F}x, \mathfrak{F}x) \end{array}$$

Example 4.27. Let $\mathscr{C}$ be a $\mathscr{V}$-category and fix an object $c \in \mathscr{C}$. The *enriched representable (covariant) $\mathscr{V}$-functor* $\mathscr{C}(c,-) \colon \mathscr{C} \to \mathscr{V}$ is defined on objects by the assignment $\mathscr{C} \ni x \mapsto \mathscr{C}(c,x) \in \mathscr{V}$ and the assignment

$$\mathscr{C}(c,-)_{x,y} \colon \mathscr{C}(x,y) \to [\mathscr{C}(c,x), \mathscr{C}(c,y)]$$

is defined by means of the adjunct of the internal composition map for $\mathscr{C}$

$$\mathscr{C}(x,y) \otimes \mathscr{C}(c,x) \xrightarrow{\quad \circ \quad} \mathscr{C}(c,y)$$

Analogously, the *enriched representable (contravariant) $\mathscr{V}$-functor* $\mathscr{C}(-,c) \colon \mathscr{C} \to \mathscr{V}$ is defined on objects by the assignment $\mathscr{C} \ni x \mapsto \mathscr{C}(x,c)$ and the assignment

$$\mathscr{C}(-,c)_{x,y} \colon \mathscr{C}(x,y) \to [\mathscr{C}(y,c), \mathscr{C}(x,c)]$$

is defined by means of the adjunct of the internal composition map

$$\mathscr{C}(y,c) \otimes \mathscr{C}(x,y) \xrightarrow{\quad \circ \quad} \mathscr{C}(x,c)$$

Definition 4.28. A *$\mathscr{V}$-enriched natural transformation* or *$\mathscr{V}$-natural transformation* $\alpha \colon \mathfrak{F} \to \mathfrak{U}$ between $\mathscr{V}$-enriched functors $\mathfrak{F}, \mathfrak{U} \colon \mathscr{C} \to \mathscr{D}$ is defined by the following data:

- For all $x \in \mathscr{C}$ an arrow $\alpha_x \colon \mathbb{1} \to \mathscr{D}(\mathfrak{F}x, \mathfrak{U}x)$ so that for each pair of objects $x, y \in \mathscr{C}$, the following square commutes in $\mathscr{V}$:

$$\begin{array}{ccc} \mathscr{C}(x,y) & \xrightarrow{\alpha_y \otimes \mathfrak{F}} & \mathscr{D}(\mathfrak{F}y, \mathfrak{U}y) \otimes \mathscr{D}(\mathfrak{F}x, \mathfrak{F}y) \\ {\scriptstyle \mathfrak{U}\otimes\alpha_x}\downarrow & & \downarrow{\scriptstyle \circ} \\ \mathscr{D}(\mathfrak{U}x, \mathfrak{U}y) \otimes \mathscr{D}(\mathfrak{F}x, \mathfrak{U}x) & \xrightarrow[\circ]{} & \mathscr{D}(\mathfrak{F}x, \mathfrak{U}y) \end{array}$$

Remark 4.29. There is an obvious composition for $\mathscr{V}$-natural transformations: the *vertical composite* $\beta\alpha$ of $\mathscr{V}$-natural transformations $\alpha \colon \mathfrak{F} \to \mathfrak{U}$ and $\beta \colon \mathfrak{U} \to \mathfrak{H}$, both from $\mathscr{C} \to \mathscr{D}$, has component $\beta\alpha)_x$ at $x \in \mathscr{C}$ defined by the composite

$$\mathbb{1} \xrightarrow{\beta_x \otimes \alpha_x} \mathscr{D}(\mathfrak{U}x, \mathfrak{H}x) \otimes \mathscr{D}(\mathfrak{F}x, \mathfrak{U}x) \xrightarrow{\circ} \mathscr{D}(\mathfrak{F}x, \mathfrak{H}x)$$

Having such a notion of coomposition, a $\mathscr{V}$-natural transformation $\alpha \colon \mathfrak{F} \to \mathfrak{U}$ is called a *$\mathscr{V}$-natural isomorphism* if there exists an inverse $\alpha^{-1} \colon \mathfrak{U} \to \mathfrak{F}$.

Example 4.30. A morphism $f \colon \mathbb{1} \to \mathscr{C}(x, y)$ in the underlying category of a $\mathscr{V}$-category $\mathscr{C}$ defines a $\mathscr{V}$-natural transformation $f^\star \colon \mathscr{C}(y, -) \to \mathscr{C}(x, -)$ whose component at $z \in \mathscr{C}$ is defined by applying the isomorphism

$$\mathscr{V}(\mathbb{1}, [\mathscr{C}(y, z), \mathscr{C}(x, z)]) \cong \mathscr{V}(\mathscr{C}(y, z), \mathscr{C}(x, z))$$

to the morphism

$$\mathbb{1} \xrightarrow{f} \mathscr{C}(x, y) \xrightarrow{\mathscr{C}(-,z)} [\mathscr{C}(y, z), \mathscr{C}(x, z)]$$

Corollary 4.31. *For any cartesian closed category $\mathscr{V}$, there is a 2-category $\mathscr{V}$-*Cat *of $\mathscr{V}$-categories, $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations.*

Proof. See [38] A.3.6. □

Lemma 4.32. *For objects x, y in a $\mathscr{V}$-category $\mathscr{C}$ the following are equivalent:*

(*i*) *x and y are isomorphic as objects of the underlying category of $\mathscr{C}$.*
(*ii*) *The* Set*-valued unenriched representable functors $\mathscr{C}_0(x, -), \mathscr{C}_0(y, -) \colon \mathscr{C} \to$* Set *are naturally isomorphic.*
(*iii*) *The $\mathscr{V}$-valued unenriched representable functors $\mathscr{C}(x, -), \mathscr{C}(y, -) \colon \mathscr{C} \to \mathscr{V}$ are naturally isomorphic.*
(*iv*) *The $\mathscr{V}$-valued $\mathscr{V}$-functors $\mathscr{C}(x, -), \mathscr{C}(y, -) \colon \mathscr{C} \to \mathscr{V}$ are $\mathscr{V}$-naturally isomorphic.*

Proof. One notes that the underlying set functor is actually a 2-functor $(-)_0 \colon \mathscr{V}$-Cat $\to$ Cat. Hence the fourth statement implies the third. The tirhd statement implies the second by whiskering with the underlying set functor $(-)_0 \colon \mathscr{V} \to$ Set. The second statement implies the first by the unenriched Yoneda Lemma. Finally, the first statement implies the last as follows: if $f \colon \mathbb{1} \to \mathscr{C}(x, y)$ and $g \colon \mathbb{1} \to \mathscr{C}(y, x)$ define an isomorphism in the underlying category of $\mathscr{C}$, then the corresponding $\mathscr{V}$-natural transformations of Example 4.30 define a $\mathscr{V}$-natural isomorphism. □

4.3.1. *Enriched (Co)Ends.* Let $\mathscr{V}$ be a closed symmetric monoidal category and let $\mathscr{C}$ be $\mathscr{V}$-enriched. Let $\mathfrak{F} \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{V}$ be a $\mathscr{V}$-enriched functor. Then there is a *covariant action* of $\mathscr{C}$ on $\mathfrak{F}$, with components

$$\zeta_{x,y,z} \colon \mathfrak{F}(x, y) \otimes \mathscr{C}(y, z) \to \mathfrak{F}(x, z)$$

as well as a *contravariant action* of $\mathscr{C}$ on $\mathfrak{F}$ with components

$$\xi_{x,y,z} \colon \mathfrak{F}(y, z) \otimes \mathscr{C}(x, y) \to \mathfrak{F}(x, z)$$

Spelling this out explicitly, the covariant action comes about by taking the adjunct of the morphism

$$\Big(\mathfrak{F}(x,-)\colon \mathscr{C}(y,z) \to [\mathfrak{F}(x,y), \mathfrak{F}(x,z)]\Big) \in \mathscr{V}(\mathscr{C}(y,z), [\mathfrak{F}(x,y), \mathfrak{F}(x,z)])$$

while the contravariant action is obtained as the adjunct of the morphism

$$\Big(\mathfrak{F}(-,z)\colon \mathscr{C}(x,y) \to [\mathfrak{F}(y,z), \mathfrak{F}(x,z)]\Big) \in \mathscr{V}(\mathscr{C}(x,y), [\mathfrak{F}(y,z), \mathfrak{F}(x,z)])$$

Definition 4.33. Let $\mathscr{C}$ be a $\mathscr{V}$-enriched category and suppose we are given a $\mathscr{V}$-enriched functor $\mathfrak{F}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{V}$,

- A *$\mathscr{V}$-extranatural transformation* $\vartheta\colon v \dot{\to} \mathfrak{F}$ from v to $\mathfrak{F}$ consists of a family of arrows in $\mathscr{V}$
$$\vartheta_c\colon v \to \mathfrak{F}(c,c)$$
indexed by objects $c \in \mathscr{C}$, such that for every pair of objects (x,y) in $\mathscr{C}$, the composites below agree:
$$v \otimes \mathscr{C}(x,y) \xrightarrow{\vartheta_x \otimes \mathrm{id}} \mathfrak{F}(x,x) \otimes \mathscr{C}(x,y) \xrightarrow{\zeta_{x,x,y}} \mathfrak{F}(x,y)$$
$$v \otimes \mathscr{C}(x,y) \xrightarrow{\vartheta_y \otimes \mathrm{id}} \mathfrak{F}(y,y) \otimes \mathscr{C}(x,y) \xrightarrow{\xi_{x,y,y}} \mathfrak{F}(x,y)$$
- A *$\mathscr{V}$-enriched end of* $\mathfrak{F}$ is an object
$$\int_{c\colon \mathscr{C}} \mathfrak{F}(c,c) \in \mathscr{V}$$
equipped with a $\mathscr{V}$-extranatural transformation
$$\vartheta\colon \int_{c\colon \mathscr{C}} \mathfrak{F}(c,c) \dot{\to} \mathfrak{F}$$
such that for any other $\mathscr{V}$-extranatural transformation $\omega\colon v \dot{\to} \mathfrak{F}$, there exists a unique morphism $f\colon v \to \int_{c\colon \mathscr{C}} \mathfrak{F}(c,c)$ such that
$$\omega_c = \vartheta_c f$$
for all objects c in $\mathscr{C}$.

Remark 4.34. For $\mathscr{V}$ a closed symmetric monoidal category and $\mathscr{C}$ a $\mathscr{V}$-enriched category along with a $\mathscr{V}$-enriched functor $\mathfrak{F}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{V}$ a $\mathscr{V}$-enriched functor, the enriched end of $\mathfrak{F}$ is equivalently given as the equalizer of

$$\prod_{c \in \mathscr{C}} \mathfrak{F}(c,c) \underset{\xi}{\overset{\zeta}{\rightrightarrows}} \prod_{c_1, c_2 \in \mathscr{C}} [\mathscr{C}(c_1,c_2), \mathfrak{F}(c_1,c_2)]$$

with ζ in components given by

$$\xi_{c_1,c_2}\colon \mathfrak{F}(c_2,c_2) \to [\mathscr{C}(c_1,c_2), \mathfrak{F}(c_1,c_2)]$$

which is defined to be the adjunct of

$$\mathfrak{F}(-,c_2)\colon \mathscr{C}(c_1,c_2) \to [\mathfrak{F}(c_2,c_2), \mathfrak{F}(c_1,c_2)]$$

Similarily, ξ has components given by

$$\zeta_{c_1,c_2}\colon \mathfrak{F}(c_1,c_1) \to [\mathscr{C}(c_1,c_2), \mathfrak{F}(c_1,c_2)]$$

which is defined to be the adjunct of

$$\mathfrak{F}(c_1,-)\colon \mathscr{C}(c_1,c_2) \to [\mathfrak{F}(c_1,c_1), \mathfrak{F}(c_1,c_2)]$$

Dually, the *enriched coend* is the coequalizer of

$$\coprod_{c_1,c_2} \mathscr{C}(c_2,c_1)\otimes \mathfrak{F}(c_1,c_2) \rightrightarrows \coprod_{c} \mathfrak{F}(c,c)$$

where the parallel morphisms are again induced by the covariant and contravariant action of $\mathfrak{F}$.

4.3.2. *Enriched Yoneda Lemma.* In order to make sense of an enriched Yoneda Lemma, we need to define enriched functor categories:

Definition 4.35. Let $\mathscr{C}$ and $\mathscr{D}$ be $\mathscr{V}$-enriched categories. Then the *$\mathscr{V}$-enriched functor category $\mathscr{D}^{\mathscr{C}}$* is the $\mathscr{V}$-enriched category whose

- objects are given by $\mathscr{V}$-enriched functors $\mathscr{C} \to \mathscr{D}$.
- hom-objects in $\mathscr{V}$ are given by the enriched end-formula:

$$\mathscr{D}^{\mathscr{C}}(\mathfrak{F},\mathfrak{U}) := \int_{c:\,\mathscr{C}} \mathscr{D}(\mathfrak{F}c,\mathfrak{U}c)$$

Lemma 4.36. *The underlying set of the $\mathscr{V}$-object of $\mathscr{V}$-natural transformations $\mathscr{V}^{\mathscr{C}}(\mathfrak{F},\mathfrak{U})$ is the set of $\mathscr{V}$-natural transformations $\mathfrak{F}\to\mathfrak{U}$.*

Proof. The underlying set functor $(-)_0 = \mathscr{V}(\mathbb{1},-)$ preserves all limits. Therefore, there is an equalizer diagram in Set of the form

$$\mathscr{V}(\mathbb{1}, \int_{c:\,\mathscr{C}} [\mathfrak{F}c,\mathfrak{U}c]) \dashrightarrow \prod_{c\in\mathscr{C}} \mathscr{V}(\mathfrak{F}c,\mathfrak{U}c) \rightrightarrows \prod_{c,c'\in\mathscr{C}} \mathscr{V}(\mathfrak{F}c',\mathfrak{U}c)$$

where we identified $\mathscr{V}(\mathbb{1},[\mathfrak{F}c,\mathfrak{U}c]) \cong \mathscr{V}(Fc,\mathfrak{U}c)$. The object in the middle is the set of indexed sets of component morphisms $\{\mathfrak{F}c \xrightarrow{\eta_c} \mathfrak{U}c\}_{c\in\mathscr{C}}$. The fact that $\mathscr{V}(\mathbb{1}, \int_{c:\,\mathscr{C}} [\mathfrak{F}c,\mathfrak{U}c])$ is an equalizer for the above parallel pair then precisely means that its elements are $\mathscr{V}$-enriched natural transformations. □

Example 4.37. For $\mathscr{V}$ = Set, the above reproduces the ordinary functor category.

Example 4.38. For $\mathscr{V} = \mathbb{R}_{\geqslant 0}\cup\{\infty\}$ with the monoidal product given by addition, a $\mathscr{V}$-enriched category X is simply a metric space, with the distance between points $x,y\in X$ given by $X(x,y)$. Given two such metric spaces X,Y and maps $f,g\colon X\to Y$, the distance between the maps is

$$Y^X(f,g) = \int_{x:\,X} Y(f(x),g(x)) = \sup_{x\in X} Y(f(x),g(x))$$

The Yoneda Lemma essentially boils down to 'evaluation at the identity is an isomorphism'. In the enriched context the enriched object of natural transformations is defined via a limit, so it is more straightforward to define the map which induces a natural transformation instead. Given an object c in a small $\mathscr{V}$-category $\mathscr{C}$ and a $\mathscr{V}$-functor $\mathfrak{F}\colon \mathscr{C}\to\mathscr{V}$, the internal action of $\mathfrak{F}$ on arrows transposes to define a map that equalizes the parallel pair

$$\mathfrak{F}c \dashrightarrow \prod_{z\in\mathscr{C}} [\mathscr{C}(c,z),\mathfrak{F}z] \rightrightarrows \prod_{x,y\in\mathscr{C}} [\mathscr{C}(c,x)\otimes\mathscr{C}(c,y),\mathfrak{F}y]$$

and thus this induces a canonical map $\mathfrak{F}c \to \mathscr{V}^{\mathscr{C}}(\mathscr{C}(c,-),\mathfrak{F})$ in $\mathscr{V}$.

Theorem 4.39 (Enriched Yoneda Lemma)**.** *For any small $\mathscr{V}$-category $\mathscr{C}$, any object $c\in\mathscr{C}$, and any $\mathscr{V}$-functor $\mathfrak{F}\colon\mathscr{C}\to\mathscr{V}$, the canonical map defines an isomorphism in $\mathscr{V}$*

$$\mathfrak{F}c \xrightarrow{\ \cong\ } \mathscr{V}^{\mathscr{C}}(\mathscr{C}(c,-),\mathfrak{F})$$

which is $\mathscr{V}$-natural in both c and $\mathfrak{F}$. In terms f enriched ends, this reads as

$$\int_{c\colon \mathscr{C}} [\mathscr{C}(c,c'), Fc'] \cong \mathfrak{F}c$$

Proof. In order to verify the isomorphism, it suffices to show that the internal action of $\mathfrak{F}$ constitutes a limit cone together with $\mathfrak{F}c$. So suppose we are given another cone over the parallel pair

$$v \xrightarrow{\quad\lambda\quad} \prod_{z\in\mathscr{C}} [\mathscr{C}(c,z), \mathfrak{F}z] \rightrightarrows \prod_{x,y\in\mathscr{C}} [\mathscr{C}(c,x)\otimes\mathscr{C}(x,y), \mathfrak{F}y]$$

We then define a candidate factorization by evaluating the transpose of the component λ_c at 1_c:

$$\lambda_c(1_c) := v \xrightarrow{1_c\otimes v} \mathscr{C}(c,c)\otimes v \xrightarrow{\lambda_c} \mathfrak{F}c$$

That $\lambda_c(1_c)\colon v \to \mathfrak{F}c$ indeed defines a factorization of λ through the limit cone, it suffices to show commutativity at each component $[\mathscr{C}(c,z), \mathfrak{F}z]$ of the product, which one verifies in transposed form:

$$\begin{array}{ccc}
\mathscr{C}(c,z)\otimes v & & \\
\downarrow{\scriptstyle \mathrm{id}\otimes 1_c\otimes v} & \searrow{=} & \\
\mathscr{C}(c,z)\otimes\mathscr{C}(c,c)\otimes v & \xrightarrow{\circ\otimes v} & \mathscr{C}(c,z)\otimes v \\
\downarrow{\scriptstyle \mathrm{id}\otimes\lambda_c} & & \downarrow{\scriptstyle \lambda_z} \\
\mathscr{C}(c,z)\otimes\mathfrak{F}z & \xrightarrow[\mathfrak{F}_{c,z}]{} & \mathfrak{F}z
\end{array}$$

The upper triangle commutes, because of the identity law for $\mathscr{C}$ while the bottom square commutes because λ defies a cone over the parallel pair. For the remaining details, see Theorem A.3.11 [38]. □

Corollary 4.40. *For any small $\mathscr{V}$-category $\mathscr{C}$, any object $c \in \mathscr{C}$ ad any $\mathscr{V}$-functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{V}$, there is a natural bijection between $\mathscr{V}$-natural transformations $\alpha\colon \mathscr{C}(c,-) \to \mathfrak{F}$ and elements $u\colon \mathbb{1} \to \mathfrak{F}c$ in the underlying set of $\mathfrak{F}c$ implemented by evaluationg the component at $c \in \mathscr{C}$ at the identity 1_c.*

Definition 4.41. A *cosmos* is a complete, cocomplete, closed symmetric monoidal category $\mathscr{V}$.

Proposition 4.42 (Enriched Co-Yoneda Lemma)**.** *Let $\mathscr{V}$ be a cosmos. For $\mathfrak{F}\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ a $\mathscr{V}$-enriched functor, and for $c \in \mathscr{C}$, there is a natural isomorphism*

$$\mathfrak{F} \cong \int^{c\colon \mathscr{C}} \mathscr{C}(c,-)\otimes\mathfrak{F}c$$

Proof. By the definition of enriched coends, enriched natural transformations of the form

$$\int^{c\colon \mathscr{C}} \mathscr{C}(c,-)\otimes\mathfrak{F}c \to \mathfrak{U}$$

are in natural bijection with systems of morphisms

$$\mathscr{C}(c,c')\otimes\mathfrak{F}c \to \mathfrak{U}c'$$

which satisfy compatibility conditions in their dependence on c and c'. By the internal hom adjunction these systems are in natural bijection to systems of the form

$$\mathfrak{F}c \to [\mathscr{C}(c, c'), \mathfrak{U}c']$$

satisfying analogous compatibility conditions. These in turn are in natural bijection with systems of morphisms

$$\mathfrak{F}c \to \mathscr{V}^{\mathscr{C}}(\mathscr{C}(c, -), \mathfrak{U})$$

natural in c. By the enriched Yoneda Lemma these systems are in natural bijection with systems of morphisms

$$\mathfrak{F}c \to \mathfrak{U}c$$

natural in c. In particular, all these identifications are also natural in $\mathfrak{U}$. Therefore, this shows that

$$\mathscr{V}^{\mathscr{C}}\Big(\int^{c\colon \mathscr{C}} \mathscr{C}(c, -) \otimes \mathfrak{F}c, -\Big) \cong \mathscr{V}^{\mathscr{C}}(\mathfrak{F}, -)$$

For further details see the Nlab page Geometry of physics Proposition 3.18. □

Proposition 4.43 ([38] Proposition A.3.3.14)**.** *Let $\mathfrak{A}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathscr{V}$ be a $\mathscr{V}$-functor so that for each $d \in \mathscr{D}$, the $\mathscr{V}$-functor $\mathfrak{A}(-, d)\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{V}$ is represented by some $\mathfrak{F}c \in \mathscr{D}$, meaning there exists a $\mathscr{V}$-natural isomorphism*

$$\mathscr{C}(c, \mathfrak{F}d) \cong \mathfrak{A}(c, d)$$

Then there is a unique way of extending the mapping $c \in \mathscr{C} \mapsto \mathfrak{F}c \in \mathscr{D}$ to a $\mathscr{V}$-fuctor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ so that the isomorphisms are $\mathscr{V}$-natural in $c \in \mathscr{C}$ as well as $d \in \mathscr{D}$.

Definition 4.44. Let $\mathscr{C}$ be a $\mathscr{V}$-enriched category. The *enriched covariant Yoneda embedding* is the enriched functor

$$\text{よ}\colon \mathscr{C} \to \mathscr{V}^{\mathscr{C}^{\mathrm{op}}}, \qquad c \mapsto \mathscr{C}(-, c)$$

Analogously, the *enriched contravariant Yoneda embedding* is the enriched functor

$$\rotatebox[origin=c]{180}{よ}\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{V}^{\mathscr{C}}, \qquad c \mapsto \mathscr{C}(c, -)$$

Definition 4.45. Let $\mathscr{C}$ and $\mathscr{D}$ be $\mathscr{V}$-enriched categories.

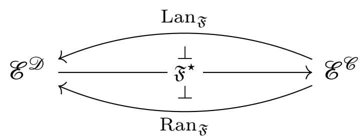
- A $\mathscr{V}$-enriched adjunction

$$\mathscr{C} \underset{\mathfrak{U}}{\overset{\mathfrak{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathscr{D}$$

  is a pair of $\mathscr{V}$-enriched functors such that we have $\mathscr{V}$-natural isomorphisms between enriched hom-functors

$$\mathscr{C}(\mathfrak{F}, -) \cong \mathscr{D}(-, \mathfrak{U})$$

- Let $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ be a functor. The *enriched left Kan extension along* $\mathfrak{F}$, denoted $\mathrm{Lan}_{\mathfrak{F}}$, is an enriched left adjoint to the precomposition functor $\mathfrak{F}^\star\colon \mathscr{E}^{\mathscr{D}} \to \mathscr{E}^{\mathscr{C}}$. Analogously, the *enriched right Kan extension along* $\mathfrak{F}$, denoted $\mathrm{Ran}_{\mathfrak{F}}$, is an enriched right adjoint to the precomposition functor $\mathfrak{F}^\star\colon \mathscr{E}^{\mathscr{D}} \to \mathscr{E}^{\mathscr{C}}$. In other words, enriched right and left Kan extensions fit into a diagram of enriched adjunctions



Analogous to standard category theory we have:

Proposition 4.46. *For $\mathscr{V}$ a cosmos, let $\mathscr{C}, \mathscr{D}$ be small $\mathscr{V}$-enriched categories and let $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ be a $\mathscr{V}$-enriched functor. Then precomposition with $\mathfrak{F}$ constitutes a $\mathscr{V}$-enriched functor*

$$\begin{aligned} \mathfrak{F}^\star\colon \mathscr{D}^{\mathscr{V}} &\to \mathscr{C}^{\mathscr{V}} \\ \mathfrak{U} &\mapsto \mathfrak{U}\mathfrak{F} \end{aligned}$$

The enriched functor $\mathfrak{F}^\star$ has both an enriched right adjoint $\mathrm{Ran}_{\mathfrak{F}}$, as well as an enriched left adjoint $\mathrm{Lan}_{\mathfrak{F}}$ given by taking right and left Kan extensions along $\mathfrak{F}$, respectively. Explicitly, the right Kan extension $\mathrm{Ran}_{\mathfrak{F}}$ evaluated at $\mathfrak{U} \in \mathscr{C}^{\mathscr{V}}$ is given by the enriched end

$$\mathrm{Ran}_{\mathfrak{F}}\mathfrak{U} \cong \int_{c\colon \mathscr{C}} [\mathscr{D}(-, \mathfrak{F}c), \mathfrak{U}c]$$

while the left Kan extension $\mathrm{Lan}_{\mathfrak{F}}$ evaluated at $\mathfrak{U} \in \mathscr{C}^{\mathscr{V}}$ is given by the enriched coend

$$\mathrm{Lan}_{\mathfrak{F}}\mathfrak{U} \cong \int^{c\colon \mathscr{C}} \mathscr{D}(\mathfrak{F}c, -) \otimes_{\mathscr{V}} \mathfrak{F}c$$

Proof. This is essentially analogous to the unenriched case. For details see the Nlab article geometry of physics – categories and toposes Proposition 3.29. □

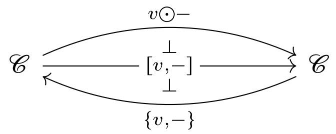
4.3.3. *Tensors and Cotensors.* This chapter is based on the Nlab page powered and copowered category.

Fix a cosmos $\mathscr{V}$.

Definition 4.47. Let $\mathscr{C}$ be a $\mathscr{V}$-enriched category.

- A *powering* or *cotensoring* of $\mathscr{C}$ over $\mathscr{V}$ is a functor $\{-,-\}\colon \mathscr{V}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ such that for any $v \in \mathscr{V}$ we have enriched natural isomorphisms (natural in $c_1, c_2 \in \mathscr{C}$)
$$[v, \mathscr{C}(c_1, c_2)] \cong \mathscr{C}(c_1, \{v, c_2\})$$
- A *copowering* or *tensoring* of $\mathscr{C}$ over $\mathscr{V}$ is a functor $\odot\colon \mathscr{V} \times \mathscr{C} \to \mathscr{C}$ such that for any $v \in \mathscr{V}$ we have enriched natural transformations (natural in $c_1, c_2 \in \mathscr{C}$)
$$\mathscr{C}(v \odot c_1, c_2) \cong [v, \mathscr{C}(c_1, c_2)]$$
- If $\mathscr{C}$ is equipped with a tensoring or cotensoring, then $\mathscr{C}$ is called *tensored* or *cotensored over* $\mathscr{V}$.

Remark 4.48. If $\mathscr{C}$ is both tensored an cotensored, then we get a pair of adjunctions



and therefore, in particular, $v \odot - \dashv \{v, -\}$.

Example 4.49. The canonical examples are given by Remark 2.21 and Remark 2.24.

Example 4.50. Suppose $\mathscr{C}$ is cocomplete. Then the category $\mathscr{C}^{\Delta^{\mathrm{op}}}$ of simplicial objects in $\mathscr{C}$ is *simplicially enriched*, i.e., sSet-enriched. Moreover, it is tensored over sSet. Indeed, the tensoring is defined by

$$-\odot-\colon \mathrm{sSet}\times\mathscr{C}^{\Delta^{\mathrm{op}}}\to\mathscr{C}^{\Delta^{\mathrm{op}}},\qquad \mathrm{sSet}\times\mathscr{C}^{\Delta^{\mathrm{op}}}\ni(S,\mathfrak{F})\mapsto\Big([n]\mapsto\coprod_{S_n}\mathfrak{F}[n]\Big)\in\mathscr{C}^{\Delta^{\mathrm{op}}}$$

The simplicial mapping object $\underline{\mathscr{C}^{\Delta^{\mathrm{op}}}}(-,-)$ may then be the deduced from the above formula as follows: We want to have a natural isomorphism

$$\mathscr{C}^{\Delta^{\mathrm{op}}}(S\odot\mathfrak{F},\mathfrak{U})\cong\mathrm{sSet}(S,\underline{\mathscr{C}^{\Delta^{\mathrm{op}}}}(\mathfrak{F},\mathfrak{U}))$$

Taking S to be representable we obtain

$$\mathscr{C}^{\Delta^{\mathrm{op}}}(\Delta^n\odot\mathfrak{F},\mathfrak{U})\cong\mathrm{sSet}(\Delta^n,\underline{\mathscr{C}^{\Delta^{\mathrm{op}}}}(\mathfrak{F},\mathfrak{U}))\cong\underline{\mathscr{C}^{\Delta^{\mathrm{op}}}}(\mathfrak{F},\mathfrak{U})_n$$

In the future we shall simply denote $\underline{\mathscr{C}^{\Delta^{\mathrm{op}}}}(\mathfrak{F},\mathfrak{U})$ by $\mathrm{Map}(\mathfrak{F},\mathfrak{U})$.
Dually, in the case where $\mathscr{C}$ is complete, the simplicially enriched category $\mathscr{C}^{\Delta^{\mathrm{op}}}$ is also cotensored:

$$\{-,-\}\colon \mathrm{sSet}^{\mathrm{op}}\times\mathscr{C}^{\Delta^{\mathrm{op}}}\to\mathscr{C}^{\Delta^{\mathrm{op}}},\qquad \mathscr{C}^{\Delta^{\mathrm{op}}}\ni(S,\mathfrak{F})\mapsto\Big([n]\mapsto\prod_{S_n}\mathfrak{F}[n]\Big)\in\mathscr{C}^{\Delta^{\mathrm{op}}}$$

4.3.4. *Day Convolution.* This chapter is based on the NLab article Day convolution and the corresponding material in [23].

Any category of functors on a (symmetric) monoidal category inherits a (symmetric) monoidal structure via a categorified convolution product. Before explaining this further, we note that there is a concept of a symmetric monidal $\mathscr{V}$-enriched category: One simply has an enriched tensor functor and suitable enriched coherence datum (for details see Definition 4.1 in [22]).

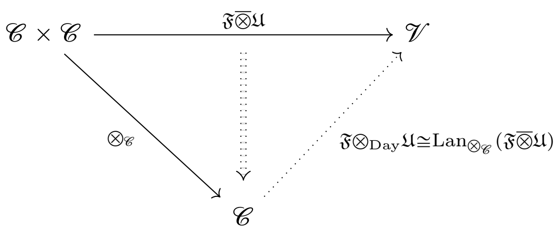
Definition 4.51. Let $\mathscr{V}$ be a closed symmetric monoidal category with all small limits and colimits and let $(\mathscr{C},\otimes,\mathbb{1})$ be a small $\mathscr{V}$-enriched monoidal category. Then the *Day convolution tensor product* on the $\mathscr{V}$-enriched functor category $\mathscr{V}^{\mathscr{C}}$

$$\begin{aligned}\otimes_{\mathrm{Day}}\colon \mathscr{V}^{\mathscr{C}}\times\mathscr{V}^{\mathscr{C}}&\to\mathscr{V}^{\mathscr{C}}\\ (\mathfrak{F},\mathfrak{U})&\mapsto\mathfrak{F}\otimes_{\mathrm{Day}}\mathfrak{U}\end{aligned}$$

is given by the enriched coend

$$(\mathfrak{F}\otimes_{\mathrm{Day}}\mathfrak{U})(c):=\int^{(c_1,c_2)\colon\mathscr{C}\times\mathscr{C}}\mathscr{C}(c_1\otimes c_2,c)\otimes_{\mathscr{V}}\mathfrak{F}c_1\otimes_{\mathscr{V}}\mathfrak{U}c_2$$

Remark 4.52. We note that if $\overline{\otimes}\colon\mathscr{V}^{\mathscr{C}}\times\mathscr{V}^{\mathscr{C}}\to\mathscr{V}^{\mathscr{C}}$ denotes the *external tensor product*, i.e., $\mathfrak{F}\overline{\otimes}\mathfrak{U}:=\otimes_{\mathscr{V}}\circ(\mathfrak{F},\mathfrak{U})$ for $\mathfrak{F},\mathfrak{U}\in\mathscr{V}^{\mathscr{C}}$, then the Day convolution product of two functors is equivalently the left Kan extension of their external tensor product along the tensor product $\otimes_{\mathscr{C}}$:



Thus, we also have the characterizing universal property given by

$$\mathscr{V}^{\mathscr{C}}(\mathfrak{F}\otimes_{\mathrm{Day}}\mathfrak{U},\mathfrak{H})\cong\mathscr{V}^{\mathscr{C}\times\mathscr{C}}(\mathfrak{F}\overline{\otimes}\mathfrak{U},\mathfrak{H}\circ\otimes_{\mathscr{C}})$$

Proposition 4.53. *For $(\mathscr{C}, \otimes, \mathbb{1})$ a small symmetric monoidal $\mathscr{V}$-enriched category, the $\mathscr{V}$-enriched functor category $\mathscr{V}^{\mathscr{C}}$ is a closed symmetric monoidal category with the tensor product being given by Day convolution, that is, $(\mathscr{V}^{\mathscr{C}}, \otimes_{\mathrm{Day}}, よ_{\mathscr{C}}\mathbb{1})$ constitutes a closed symmetric monoidal category. Its internal hom $[-,-]_{\mathrm{Day}}$ is given by the end*

$$\begin{aligned}[X,Y]_{\mathrm{Day}}(c) &\cong \int_{c_1 : \mathscr{C}} [X(c_1), Y(c \otimes c_1)]_{\mathscr{V}} \\ &\cong \int_{c_1, c_2} [\mathscr{C}(c \otimes c_1, c_2), [Xc_1, Yc_2]_{\mathscr{V}}]_{\mathscr{V}}\end{aligned}$$

where $[-,-]_{\mathscr{V}}$ is the internal hom in $\mathscr{V}$.

Proof. Let us start by verifying associativity: For $X, Y, Z \in \mathscr{V}^{\mathscr{C}}$ we have

$$\begin{aligned}X \otimes_{\mathrm{Day}} (Y \otimes_{\mathrm{Day}} Z) &\cong \int^{a,b} \mathscr{C}(a \otimes b, -) \otimes_{\mathscr{V}} Xa \otimes_{\mathscr{V}} (Y \otimes_{\mathrm{Day}} Z)(b) \\ &\cong \int^{a,b} \mathscr{C}(a \otimes b, -) \otimes_{\mathscr{V}} Xa \otimes_{\mathscr{V}} \int^{c,d} \mathscr{C}(c \otimes d, b) \otimes_{\mathscr{V}} Yc \otimes_{\mathscr{V}} Zd \\ &\cong \int^{a,b,c,d} \mathscr{C}(a \otimes b, -) \otimes_{\mathscr{V}} \mathscr{C}(c \otimes d, b) \otimes_{\mathscr{V}} Xa \otimes_{\mathscr{V}} Yc \otimes_{\mathscr{V}} Zd \\ &\cong \int^{a,c,d} \mathscr{C}(a \otimes c \otimes d, -) \otimes_{\mathscr{V}} Xa \otimes_{\mathscr{V}} Yc \otimes_{\mathscr{V}} Zd\end{aligned}$$

In the same fashion one verifies

$$(X \otimes_{\mathrm{Day}} Y) \otimes_{\mathrm{Day}} Z \cong \int^{a,c,d} \mathscr{C}(a \otimes c \otimes d, -) \otimes_{\mathscr{V}} Xa \otimes_{\mathscr{V}} Yc \otimes_{\mathscr{V}} Zd$$

Moreover, we have

$$\begin{aligned}X \otimes_{\mathrm{Day}} よ_{\mathscr{C}}\mathbb{1} &\cong \int^{a,b} \mathscr{C}(a \otimes b, -) \otimes_{\mathscr{V}} Xa \otimes \mathscr{C}(\mathbb{1}, b) \\ &\cong \int^{a,b} \mathscr{C}(a \otimes b, -) \otimes_{\mathscr{V}} \mathscr{C}(\mathbb{1}, b) \otimes Xa \\ &\cong \int^{a} \mathscr{C}(a \otimes \mathbb{1}, -) \otimes_{\mathscr{V}} Xa \\ &\cong \int^{a} \mathscr{C}(a, -) \otimes_{\mathscr{V}} Xa \\ &\cong X\end{aligned}$$

The other claims concerning the symmetric monoidal structure present similar exercises in the spirit of the usual coend yoga. Finally, let us verify the claim concerning the internal hom:

$$\begin{aligned}\mathscr{V}^{\mathscr{C}}(X \otimes_{\mathrm{Day}} Y, Z) &\cong \int_{c} \mathscr{V}((X \otimes_{\mathrm{Day}} Y)(c), Zc) \\ &\cong \int_{a,b,c} \mathscr{V}(\mathscr{C}(a \otimes b, c) \otimes_{\mathscr{V}} Xa \otimes_{\mathscr{V}} Yb, Zc)\end{aligned}$$

$$
\begin{aligned}
&\cong \int_{a,b,c} \mathscr{V}(Xa, [Yb, [\mathscr{C}(a \otimes b, c), Zc]_{\mathscr{V}}]_{\mathscr{V}}) \\
&\cong \int_{a,b} \mathscr{V}(Xa, [Yb, \int_{c} [\mathscr{C}(a \otimes b, c), Zc]_{\mathscr{V}}]_{\mathscr{V}}) \\
&\cong \int_{a,b} \mathscr{V}(Xa, [Yb, Z(a \otimes b)]_{\mathscr{V}}) \\
&\cong \int_{a} \mathscr{V}(Xa, \int_{b} [Yb, Z(a \otimes b)]_{\mathscr{V}}) \\
&\cong \int_{a} \mathscr{V}(Xa, [Y, Z]_{\mathrm{Day}}(a)) \\
&\cong \mathscr{V}^{\mathscr{C}}(X, [Y, Z]_{\mathrm{Day}})
\end{aligned}
$$

Showing the second formula for the internal hom is an easy exercise in the coend yoga. ☐

Remark 4.54. Let $\mathscr{C}$ be as in the above proposition and consider the Yoneda embeddings $\text{よ}c$ and $\text{よ}c'$ for objects $c, c' \in \mathscr{C}$. Then the Day convolution tensor product of representables is the representable of the respective tensor product of objects:

$$
\begin{aligned}
(\text{よ}c \otimes_{\mathrm{Day}} \text{よ}c')(\tilde{c}) &= \int^{c_1, c_2} \mathscr{C}(c_1 \otimes c_2, \tilde{c}) \otimes_{\mathscr{V}} \text{よ}c(c_1) \otimes_{\mathscr{V}} \text{よ}c'(c_2) \\
&\cong \int^{c_2} \int^{c_1} \Big(\mathscr{C}(c_1 \otimes c_2, \tilde{c}) \otimes_{\mathscr{V}} \text{よ}c(c_1) \Big) \otimes_{\mathscr{V}} \text{よ}c'(\tilde{c}) \\
&\cong \int^{c_2} \mathscr{C}(c \otimes c_2, \tilde{c}) \otimes_{\mathscr{V}} \text{よ}c'(\tilde{c}) \\
&\cong \text{よ}(c \otimes c')(\tilde{c})
\end{aligned}
$$

In that sense, Day convolution is an extension of the ordinary tensor product to the whole functor category and not just its representables.

Example 4.55. Let $\mathscr{V} := \mathrm{Set}$ and let $\mathscr{C}$ be a category with finite products and terminal object, that is, a cartesian symmetric monoidal category. Now consider the presheaf category $\mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$ which, by Proposition 4.53, may be endowed with a closed symmetric monoidal structure given by the Day convolution tensor product, since $\mathscr{C}^{\mathrm{op}}$ is a $\mathrm{Set}^{\mathrm{op}}$-enriched category. For $X, Y \in \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$, we have

$$
\begin{aligned}
X \otimes Y &:= \int^{c_1, c_2} \mathscr{C}^{\mathrm{op}}(c_1 \times^{\mathrm{op}} c_2, -) \times Xc_1 \times Yc_2 \\
&\cong \int^{c_1, c_2} \text{よ}c_1 \times \text{よ}c_2 \times Xc_1 \times Yc_2 \\
&\cong \Big(\int^{c_1} \text{よ}c_1 \times Xc_1 \Big) \times \Big(\int^{c_2} \text{よ}c_2 \times Yc_2 \Big) \\
&\cong X \times Y
\end{aligned}
$$

Thus in this case the Day convolution monoidal structure agrees with the typical cartesian monoidal structure on the presheaf category $\mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$. The corresponding internal hom may then also be deduced by using the second formula presented in

Proposition 4.53:

$$
\begin{aligned}
[X,Y]_{\mathrm{Day}}(c) &= \int_{c_1,c_2} \mathrm{Set}(\mathscr{C}^{\mathrm{op}}(c\times c_1, c_2), \mathrm{Set}(Xc_1, Yc_2)) \\
&\cong \int_{c_1,c_2} \mathrm{Set}(\mathscr{C}(c_2,c)\times \mathscr{C}(c_2,c_1),\ \mathrm{Set}(Xc_1,Yc_2)) \\
&\cong \int_{c_1,c_2} \mathrm{Set}(\mathscr{C}(c_2,c)\times \mathscr{C}(c_2,c_1)\times Xc_1,\ Yc_2) \\
&\cong \int_{c_2} \mathrm{Set}\Big(\mathscr{C}(c_2,c)\times \Big(\int^{c_1} \mathscr{C}(c_2,c_1)\times Xc_1\Big),\ Yc_2\Big) \\
&\cong \int_{c_2} \mathrm{Set}(\mathscr{C}(c_2,c)\times Xc_2, Yc_2) \\
&\cong \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}(よ c\times X, Y)
\end{aligned}
$$

which retrieves the formula from Example 4.14.

Example 4.56. Consider the category $\mathscr{F}$ of finite pointed sets, which has as its objects finite sets which have chosen basepoints. Morphisms between such pointed sets are given by functions between the sets that map basepoint to basepoint. The category $\mathscr{F}$ is symmetric monoidal with the smash product

$$\wedge\colon \mathscr{F}\times\mathscr{F}\to\mathscr{F}, \qquad (F_1,F_2)\mapsto F_1\wedge F_2 := \frac{F_1\times F_2}{F_1\vee F_2}$$

where $\vee\colon \mathscr{F}\times\mathscr{F}\to\mathscr{F}$ is the wedge sum, which maps (F_1,F_2) to the quotient of the disjoint union of F_1 and F_2 where the respective basepoints are identified:

$$\frac{F_1\coprod F_2}{\star_{F_1}\sim\star_{F_2}}$$

To summarize and to put it more concretely, the smash product of two pointed sets F_1 and F_2 is the quotient of the cartesian product $F_1\times F_2$, where all points with the basepoint as a coordinate are identified (that is, $(\star_{F_1}, f_2)\sim\star\sim(f_1,\star_{F_2})$ for all $f_1\in F_1$ and all $f_2\in F_2$). In particular, if $F_1 := \{\star,1,\dots,l\}$ and $F_2 := \{\star,1,\dots,l'\}$ for natural numbers l,l', then $F_1\wedge F_2 = \{\star,1,\dots,ll'\}$. In fact, the symmetric monoidal category $\mathscr{F}$ is closed:

$$\mathscr{F}(F_1\wedge F_2, F_3)\cong\mathscr{F}(F_1, F_3^{F_1})$$

where $F_3^{F_1}$ is the corresponding internal hom which is given as the pointed set of basepoint preserving functions $F_1\to F_3$, which itself has as the distinguished basepoint the constant function $\star\colon F_1\to F_3$ which maps every $f\in F_1$ to the basepoint in F_3. Moreover, consider the simplex category Δ, which may also be viewed as a symmetric monoidal category, where the corresponding tensor functor $+\colon\Delta\times\Delta\to\Delta$ is simply given by taking the product in the category Δ. Note that a product of $[n]$ and $[m]$ in Δ is given by $[n+m]$ with the evident projection maps. Finally, let Cart be the category of *cartesian spaces*, which has as its set of objects all those open subsets $U\subset\mathbb{R}^d$, for varying d, such that U is diffeomorphic to $\mathbb{R}^d$. Morphisms in Cart are given by smooth maps between the respective open subsets. This category is symmetric monoidal too by means of the cartesian product.

Having all this, we let $\mathscr{V} := \mathrm{Set}$ and $\mathscr{C} := (\Delta^{\times d})^{\mathrm{op}}\times\mathscr{F}\times\mathrm{Cart}^{\mathrm{op}}$, where $\Delta^{\times d} := \prod_{i=1}^{d}\Delta$

is the n-fold product of the simplex category. The category $\mathscr{C}$ is then symmetric monoidal and $\mathscr{V}$-enriched (just a standard category). By Proposition 4.53 we thus obtain that the functor category $\mathscr{V}^{\mathscr{C}}$ is a closed symmetric monoidal category with the tensor product being given by Day convolution, that is, for $X, Y \in \mathscr{V}^{\mathscr{C}}$ the Day convolution tensor product $X \otimes Y$ is given by

$$\int^{\mathscr{C}}\int^{\mathscr{C}} \Delta^{\times d}(\mathbf{m}, \mathbf{m}_1 + \mathbf{m}_2) \times \mathscr{F}(F_1 \wedge F_2, F) \times \mathrm{Cart}(U, U_1 \times U_2) \times X(\mathbf{m}_1, F_1, U_1) \times Y(\mathbf{m}_2, F_2, U_2)$$

where the coend is taken over the variables $(\mathbf{m}_i, F_i, U_i) \in \Delta^{\times d} \times \mathscr{F} \times \mathrm{Cart}$ for $i = 1, 2$. Motivation for why we would choose $\mathscr{C}$ the way we did here is given in later sections on *smooth symmetric monoidal* (∞, d)*-categories*.

5. Model Categories

Frodo: 'I wonder what sort of a tale we've fallen into?'
Sam: 'I wonder if we'll ever be put into songs or tales.'
Frodo: 'What? You mean like in the songs where they tell you what to do?'
Sam: 'No, I mean the ones that really mattered. Full of darkness and danger, they were, and sometimes you didn't want to know the end, because how could the end be happy? How could the world go back to the way it was when so much bad had happened? But in the end, it's only a passing thing, this shadow. Even darkness must pass. A new day will come, and when the sun shines, it'll shine out the clearer.

Tolkien, J.R.R. The Fellowship of the Ring

The following Chapter is based on [19].

The category of topological spaces naturally allows us to study continuous deformations between spaces, which is captured by the notion of a homotopy. Model category theory is a powerful framework that generalizes homotopy theory by introducing an abstract category with additional structure that captures the notion of homotopy between morphisms. A model category provides a unified approach to many different areas of homotopy theory and allows us to define homotopy limits and colimits, as well as notions of homotopy equivalences between objects. This extra structure is essential for studying the homotopical behavior of mathematical objects in a wide range of contexts, including algebraic topology, algebraic geometry, and algebraic K-theory. This chapter will introduce the basic concepts and properties of model categories, and lay the groundwork for future discussions of higher category theory and ∞-categories. Model category theory provides a powerful tool for understanding the homotopical structure of mathematical objects and their relationships to one another, and is a key ingredient in many areas of modern mathematics and physics.

5.1. **Definitions.** Recall that $\mathscr{C}^{[n]}$ is the category of functors $[n] \to \mathscr{C}$. In particular, if $n = 1$ the category $\mathscr{C}^{[1]}$ may be identified with the category that has as objects morphisms of $\mathscr{C}$ and as morphisms commutative squares in $\mathscr{C}$.

Definition 5.1. Let $\mathscr{C}$ be a category.

- A morphism f in $\mathscr{C}$ is called a *retract* of a morphism f' in $\mathscr{C}$, if there exists a commutative diagram

$$\begin{array}{ccccc} & & \overset{1_{\mathrm{dom}f}}{=\!=\!=\!=\!=\!=\!=\!=\!=} & & \\ \mathrm{dom}f & \longrightarrow & \mathrm{dom}f' & \longrightarrow & \mathrm{dom}f \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{cod}f & \longrightarrow & \mathrm{cod}f' & \longrightarrow & \mathrm{cod}f \\ & & \underset{1_{\mathrm{cod}f}}{=\!=\!=\!=\!=\!=\!=\!=\!=} & & \end{array}$$

- A *functorial factorization* for $\mathscr{C}$ is a section $\Xi\colon \mathscr{C}^{[1]} \to \mathscr{C}^{[2]}$ of the composition functor $d_1\colon \mathscr{C}^{[2]} \to \mathscr{C}^{[1]}$, i.e., $d_1\Xi = 1_{\mathscr{C}^{[1]}}$.

Remark 5.2. Equivalently, a functorial factorization is a pair (Ξ_1, Ξ_2) with Ξ_1, Ξ_2 being functors $\mathscr{C}^{[1]} \to \mathscr{C}^{[1]}$ such that for any morphism f in $\mathscr{C}$ we have the following factorization:

$$\begin{array}{ccc} \mathrm{dom}f & \xrightarrow{\quad f \quad} & \mathrm{cod}f \\ & {\scriptstyle \Xi_1 f}\searrow \quad \nearrow{\scriptstyle \Xi_2 f} & \\ & X & \end{array}$$

Definition 5.3. Let i and p be morphisms in $\mathscr{C}$. We say that i has the *left lifting property (LLP)* with respect to p and p has the *right lifting property (RLP)* with respect to i, if for any commutative diagram

$$\begin{array}{ccc} \mathrm{dom}(i) & \longrightarrow & \mathrm{dom}(p) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{cod}(i) & \longrightarrow & \mathrm{cod}(p) \end{array}$$

there exists a lift $h\colon \mathrm{cod}(i) \to \mathrm{dom}(p)$ such that

$$\begin{array}{ccc} \mathrm{dom}(i) & \longrightarrow & \mathrm{dom}(p) \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ \mathrm{cod}(i) & \longrightarrow & \mathrm{cod}(p) \end{array}$$

commutes.

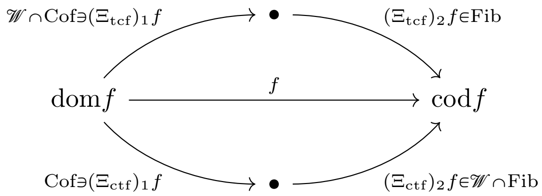
Definition 5.4. Let $\mathscr{C}$ be a category.

- A *model structure* on $\mathscr{C}$ is a triple

$$(\mathscr{W},\ \mathrm{Fib},\ \mathrm{Cof})$$

consisting of distinguished classes of morphisms $\mathscr{W}, \mathrm{Fib}, \mathrm{Cof}$ of $\mathscr{C}$ such that the following axioms hold:
  - *2-out-of 3 axiom:* The class $\mathscr{W}$ contains all isomorphisms in $\mathscr{C}$ and for all morphisms f, f' in $\mathscr{C}$, if any two of f, f', ff' is in $\mathscr{W}$, then the third is also in $\mathscr{W}$.
  - *Retract axiom:* If f, f' are morphisms in $\mathscr{C}$ such that f is a retract of f' and f' is a morphism in one of the three classes $\mathscr{W}, \mathrm{Fib}, \mathrm{Cof}$, then so is f.
  - *Lifting axiom:* The class of morphisms $\mathscr{W} \cap \mathrm{Fib}$ enjoys the LLP with respect to the morphisms in Fib. The class of morphisms $\mathscr{W} \cap \mathrm{Cof}$ enjoys the LLP with respect to the morphisms in Cof.

  – *Factorization axiom:* There exist functorial factorizations Ξ_{ctf}, Ξ_{tcf} such that for all $f \in \mathscr{C}^{[1]}$ we have



  where $\bullet$ denotes the respective (most likely distinct) codomains of the morphisms $(\Xi_{\mathrm{ctf}})_1 f$ and $(\Xi_{\mathrm{tcf}})_1 f$.

- For a model structure

$$(\mathscr{W},\ \mathrm{Fib},\ \mathrm{Cof})$$

  on $\mathscr{C}$ the morphisms in $\mathscr{W}$ are referred to as *weak equivalences*, the morphisms in Fib are called *fibrations* and the morphisms in Cof are called *cofibrations*. Morphisms in $\mathrm{Fib}^{\simeq} := \mathscr{W} \cap \mathrm{Fib}$ are called *trivial fibrations*, while the morphisms in $\mathrm{Cof}^{\simeq} := \mathscr{W} \cap \mathrm{Cof}$ are referred to as *trivial cofibrations*.
- A *model category* $\mathscr{C}$ is the information of a model structure

$$(\mathscr{C},\ \mathscr{W},\ \mathrm{Fib},\ \mathrm{Cof})$$

  on $\mathscr{C}$ such that $\mathscr{C}$ is both *complete* and *cocomplete*.

Remark 5.5. The notation Ξ_{ctf} and Ξ_{tcf} is chosen so as to remind the reader that the first functorial factorization factorizes morphisms into

$$\bullet \xrightarrow[\mathrm{c}]{\text{cofibration}} \bullet \xrightarrow[\mathrm{tf}]{\text{trivial fibration}} \bullet$$

while the second one factorizes as

$$\bullet \xrightarrow[\mathrm{tc}]{\text{trivial cofibration}} \bullet \xrightarrow[\mathrm{f}]{\text{fibration}} \bullet$$

which motivates the shorthand-notation ctf and tcf.

Example 5.6. Let $\mathscr{C}$ be a complete and cocomplete category. One can define three different model structures on $\mathscr{C}$ by defining one of the subcategories $\mathscr{W}, \mathrm{Cof}, \mathrm{Fib}$ to contain all isomorphisms of $\mathscr{C}$, and the other two to contain all maps of $\mathscr{C}$. This can be shown to give rise to three distinct model structures for $\mathscr{C}$. For details see [19].

Example 5.7. For model categories $\mathscr{B}$ and $\mathscr{C}$, the category $\mathscr{B} \times \mathscr{C}$ inherits a model structure and may thus be interpreted as a model category.

Remark 5.8. Model categories are self dual: For a model category $\mathscr{C}$ the opposite category $\mathscr{C}^{\mathrm{op}}$ has a canonical model structure induced by the one on $\mathscr{C}$. Since we also have $(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}} = \mathscr{C}$ as model categories, every theorem about model categories has a dual theorem.

Since a model category $\mathscr{C}$ is, by definition, both complete and cocomplete it must always have both a terminal object $\star \in \mathscr{C}$ and an initial object $\varnothing \in \mathscr{C}$.

Definition 5.9. Let $\mathscr{C}$ be a model category with terminal object $\star$ and initial object $\varnothing$.

- An object $X \in \mathscr{C}$ is *cofibrant* if the morphism

$$\varnothing \longrightarrow X$$

  is a cofibration.

- An object $X \in \mathscr{C}$ is *fibrant* if the morphism

$$X \longrightarrow \star$$

is a fibration.

Proposition 5.10. *Over and under categories of model categories come endowed with a model structure. More concretely, if $\mathscr{C}$ is a model category and $X \in \mathscr{C}$, then $\mathscr{C}$ induces a model structure on both $\mathscr{C}/X$ and $X/\mathscr{C}$.*

Proof Sketch. Let $\Pi\colon \mathscr{C}/X \to \mathscr{C}$ be the forgetful functor. A morphism $f \in \mathscr{C}/X$ is defined to be a cofibration (fibration, weak equivalence) if and only if Πf is a cofibration (fibration, weak equivalence). □

Note that we can decompose the morphism $\varnothing \longrightarrow X$ by means of our functorial factorization as

$$\varnothing \xrightarrow{(\Xi_{\mathrm{ctf}})_1(\varnothing\to X)} LX \xrightarrow{(\Xi_{\mathrm{ctf}})_2(\varnothing\to X)} X$$

First of all, this gives rise to a functor $L\colon \mathscr{C} \to \mathscr{C}$ which sends a morphism f in $\mathscr{C}$ to the bottom map of the image of the commutative diagram

$$\begin{array}{ccc} \varnothing & \longrightarrow & \varnothing \\ \downarrow & & \downarrow \\ \mathrm{dom}f & \xrightarrow[f]{} & \mathrm{cod}f \end{array}$$

under $(\Xi_{\mathrm{ctf}})_1$. In particular, we note that LX is cofibrant (the map $\varnothing \to LX$ is a cofibration by construction). Moreover, we get a natural transformation

$$LX \xrightarrow{l_X} X \in \mathrm{Fib}^{\simeq}$$

with $l_X = (\Xi_{\mathrm{ctf}})_2(\varnothing \to X)$. The maps $l_X\colon LX \to X$ assemble into a *natural weak equivalence*

$$l\colon L \xrightarrow{\simeq} 1_{\mathscr{C}}$$

that is, l is a natural transformation such that each component is a weak equivalence. Naturality is obtained from

$$(\Xi_{\mathrm{ctf}})_2\left(\begin{array}{ccc} \varnothing & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \varnothing & \longrightarrow & Y \end{array}\right) = \begin{array}{ccc} LX & \xrightarrow{l_X} & X \\ {\scriptstyle Lf}\downarrow & & \downarrow f \\ LY & \xrightarrow[l_Y]{} & Y \end{array}$$

Utilizing the other functorial factorization Ξ_{tcf} in the definition of the model category $\mathscr{C}$, we also obtain a functor $R\colon \mathscr{C} \to \mathscr{C}$ such that RX is fibrant and a natural weak transformation $r\colon 1_{\mathscr{C}} \xrightarrow{\simeq} R$. The functor $L\colon \mathscr{C} \to \mathscr{C}$ is referred to, very aptly so, as the *cofibrant replacement functor* and the functor $R\colon \mathscr{C} \to \mathscr{C}$ is referred to as the *fibrant replacement functor*.

Example 5.11. The most important model category we will consider is the Quillen model structure on simplicial sets $\mathrm{sSet}_{\mathrm{Quillen}}$. Cofibrations are given by monomorphisms, and weak equivalences are given by weak homotopy equivalences i.e. morphisms which become weak homotopy equivalences in Top after applying geometric realization. A fibrant replacement functor for $\mathrm{sSet}_{\mathrm{Quillen}}$ is given by Kan's functor Ex^{∞}. We will talk more about the Quillen model structure later.

Lemma 5.12 (Retract argument)**.** *Suppose $f = pi$ in a category $\mathscr{C}$, and suppose that f has the LLP with respect to p. Then f is a retract of i. Dually, if f has the RLP with respect to i, then f is a retract of p.*

Proof. Suppose f has the LLP with respect to p. Then we have a lift $r\colon \mathrm{cod} f \to \mathrm{cod}(i)$ such that

$$\begin{array}{ccc} \mathrm{dom} f & \xrightarrow{\ i\ } & \mathrm{cod}(i) \\ {\scriptstyle f}\downarrow & \overset{r}{\nearrow} & \downarrow{\scriptstyle p} \\ \mathrm{cod} f & = & \mathrm{cod}(p) \end{array}$$

commutes. But then the diagram

$$\begin{array}{ccccc} \mathrm{dom} f & = & \mathrm{dom}(i) & = & \mathrm{dom} f \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ \mathrm{cod} f & \xrightarrow[r]{} & \mathrm{cod}(i) & \xrightarrow[p]{} & \mathrm{cod} f \end{array}$$

realizes f as a retract of i. □

The retract argument from above implies that some of the axioms for a model category are redundant.

Notation 5.13. If $\mathscr{C}$ is a model category and $\mathscr{D} \subset \mathrm{Mor}\mathscr{C}$ is a family of morphisms from $\mathscr{C}$, then define ${}^{\boxslash}\mathscr{D}$ to be all those morphisms that enjoy the LLP with respect to all morphisms in $\mathscr{D}$. Analogously, $\mathscr{D}^{\boxslash}$ is the family of morphisms that enjoy the RLP with respect to all morphisms in $\mathscr{D}$.

Lemma 5.14. *Let $\mathscr{C}$ be a model category. Then the following holds:*

$$\begin{aligned} \mathrm{Cof} &= {}^{\boxslash}\mathrm{Fib}^{\simeq} \\ \mathrm{Cof}^{\simeq} &= {}^{\boxslash}\mathrm{Fib} \\ \mathrm{Fib} &= \mathrm{Cof}^{\simeq\boxslash} \\ \mathrm{Fib}^{\simeq} &= \mathrm{Cof}^{\boxslash} \end{aligned}$$

Proof. By definition of a model structure, any cofibration has the LLP with respect to trivial fibrations. Conversely, suppose f has the LLP with respect to all trivial fibrations. By means of our functorial factorizations we may factorize $f = pi$, where i is a cofibration and p is a trivial fibration. By assumption f has the LLP with respect to p, and therefore by Lemma 5.12 f is a retract of i. By the retract axiom of model categories $f \in \mathrm{Cof}$. The part with the trivial cofibration is analogous and the remaining claims follow by duality. □

Remark 5.15. In particular, any isomorphism in $\mathscr{C}$ is a trivial cofibration and a trivial fibration. Indeed, if f is an isomorphism in $\mathscr{C}$ and $f' \in \mathrm{Fib}$ such that we have a commutative square

$$\begin{array}{ccc} \mathrm{dom} f & \longrightarrow & \mathrm{dom} f' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathrm{cod} f & \longrightarrow & \mathrm{cod} f' \end{array}$$

then there is a lift $B \to C$ given by the composition

$$\mathrm{cod} f \xrightarrow{\ f^{-1}\ } \mathrm{dom} f \longrightarrow \mathrm{dom} f'$$

Corollary 5.16. *Let $\mathscr{C}$ be a model category. Then cofibrations (trivial cofibrations) are closed under pushouts. That is, if we have a pushout square*

$$\begin{array}{ccc} \mathrm{dom} f & \longrightarrow & \mathrm{dom} f' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathrm{cod} f & \longrightarrow & \mathrm{cod} f' \end{array}$$

where f is a cofibration (trivial cofibration), then f' is a cofibration (trivial cofibration). Dually, fibrations (trivial fibrations) are closed under pullbacks.

Proof. Suppose we have a pushout square

$$\begin{array}{ccc} \operatorname{dom} f & \longrightarrow & \operatorname{dom} f' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \operatorname{cod} f & \longrightarrow & \operatorname{cod} f' \end{array}$$

with f being a cofibration. By Lemma 5.14 it suffices to show that f' has the LLP with respect to all trivial fibrations. Let p be a trivial fibration in $\mathscr{C}$ and consider the lifting problem

$$\begin{array}{ccccc} \operatorname{dom} f & \longrightarrow & \operatorname{dom} f' & \xrightarrow{q} & \operatorname{dom}(p) \\ {\scriptstyle f}\downarrow & & & \nearrow & \downarrow{\scriptstyle p} \\ \operatorname{cod} f & \longrightarrow & \operatorname{cod} f' & \longrightarrow & \operatorname{cod}(p) \end{array}$$

which admits a lift $h\colon \operatorname{cod} f \to \operatorname{dom}(p)$, since f is a cofibration. The data of the pushout $\operatorname{cod} f'$ of the diagram $\operatorname{cod} f \xleftarrow{f} \operatorname{dom} f \longrightarrow \operatorname{dom} f'$ may be equivalently described as an initial object

$$\Big(\operatorname{cod} f',\ \lambda\colon \mathfrak{F} \to \operatorname{cod} f'\Big)$$

in the category of elements of $\mathfrak{F}$, where $\mathfrak{F}$ is the functor associated with the given pushout. The commutative diagram

$$\begin{array}{ccc} \operatorname{dom} f & \longrightarrow & \operatorname{dom} f' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ \operatorname{cod} f & \xrightarrow[h]{} & \operatorname{dom}(p) \end{array}$$

gives rise to an object

$$\Big(\operatorname{dom}(p),\ \mu\colon \mathfrak{F} \to \operatorname{dom}(p)\Big) \in \mathbf{el}(\mathfrak{F})$$

By the universal property of the pushout, we obtain the existence and uniqueness of a morphism $\varphi\colon \operatorname{cod} f' \to \operatorname{dom}(p)$ such that, in particular,

$$\begin{array}{ccc} \operatorname{dom} f' & \xrightarrow{f'} & \operatorname{cod} f' \\ {\scriptstyle q}\downarrow & \swarrow{\scriptstyle \varphi} & \\ \operatorname{dom}(p) & & \end{array}$$

commutes, but this exactly solves the lifting problem

$$\begin{array}{ccc} \operatorname{dom} f' & \xrightarrow{q} & \operatorname{dom}(p) \\ {\scriptstyle f'}\downarrow & \nearrow{\scriptstyle \varphi} & \downarrow{\scriptstyle p} \\ \operatorname{cod} f' & \longrightarrow & \operatorname{cod}(p) \end{array}$$

Showing that trivial cofibrations are closed under pushouts is analogous and the remainder follows by duality. $\square$

One of the most useful results about model categories is provided by the next lemma:

Lemma 5.17 (Ken Brown's lemma)**.** *Let $\mathscr{C}$ be a model category and suppose $\mathscr{D}$ is a category with a subcategory of weak equivalences which satisfies the* 2*-out-of* 3 *axiom.*

- *If $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ takes trivial cofibrations between cofibrant objects to weak equivalences, then $\mathfrak{F}$ takes all weak equivalences between cofibrant objects to weak equivalences.*
- *If $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ takes trivial fibrations between fibrant objects to weak equivalences, then $\mathfrak{F}$ takes all weak equivalences between fibrant objects to weak equivalences.*

Proof. Suppose we are given a weak equivalence f of cofibrant objects. By means of the universal property of the coproduct we may define a map $(f, 1_{\mathrm{cod}f})\colon \mathrm{dom}f \coprod \mathrm{cod}f \to \mathrm{cod}f$. Factor this map into a cofibration $\mathrm{dom}f \coprod \mathrm{cod}f \xrightarrow{q} C$ followed by a trivial fibration $C \xrightarrow{p} \mathrm{cod}f$. Considering the pushout diagram

$$\begin{array}{ccc} \varnothing & \longrightarrow & \mathrm{dom}f \\ \big\downarrow & & \big\downarrow{\scriptstyle \iota_{\mathrm{dom}f}} \\ \mathrm{cod}f & \xrightarrow[\iota_{\mathrm{cod}f}]{} & \mathrm{dom}f \coprod \mathrm{cod}f \end{array}$$

shows that the inclusion maps $\mathrm{dom}f \xrightarrow{\iota_{\mathrm{dom}f}} \mathrm{dom}f \coprod \mathrm{cod}f$ and $\mathrm{cod}f \xrightarrow{\iota_{\mathrm{cod}f}} \mathrm{dom}f \coprod \mathrm{cod}f$ are cofibrations. Since $pq\iota_{\mathrm{dom}f} = f$ and p are weak equivalences, by the 2-out-of 3 axiom, $q\iota_{\mathrm{dom}f}$ is a weak equivalence. Analogously, $q\iota_{\mathrm{cod}f}$ is a weak equivalence and hence both $q\iota_{\mathrm{dom}f}$ and $q\iota_{\mathrm{cod}f}$ are trivial cofibrations (of cofibrant objects). By assumption, both $\mathfrak{F}(q\iota_{\mathrm{dom}f})$ and $\mathfrak{F}(q\iota_{\mathrm{cod}f})$ are weak equivalences. Since $\mathfrak{F}(pq\iota_{\mathrm{cod}f}) = \mathfrak{F}(1_{\mathrm{cod}f})$ is also a weak equivalence, we conclude from the 2-out-of 3 axiom that $\mathfrak{F}(p)$ is a weak equivalence, and hence that $\mathfrak{F}(f) = \mathfrak{F}(pq\iota_{\mathrm{dom}f})$ is a weak equivalence, as claimed. The dual statement follows analogously. □

5.2. **The Homotopy Category.** A bold category theorist, or maybe a delusional one for that matter, would sometimes like to consider a morphism f in a general category $\mathscr{C}$ as a grand generalization of a path from $\mathrm{dom}f$ to $\mathrm{cod}f$, somehow presupposing the notion of space which allows for such ideas. A formal (or symbolic) zig zag of such (composable) paths $f_1, \dots, f_n$ is then nothing else than an object f in the functor category $\mathscr{C}^{[n]}$. As the name of this chapter might suggest, we need to bring in a notion of homotopy theory here. A category is said to be equipped with a subcategory of *weak equivalences* if this subcategory has the same objects (it is wide) and its class of morphisms satisfies the 2-out-of 3 axiom. Assuming the existence of such a subcategory $\mathscr{W}$ for the category $\mathscr{C}$ and thinking about the morphisms in $\mathscr{W}$ as weak equivalences between spaces, we might get to the idea of formally inverting the arrows in $\mathscr{W}$.

Definition 5.18. Suppose $\mathscr{C}$ is a category with a subcategory of weak equivalences $\mathscr{W}$.

- For $n \in \mathbb{N}$, the collection of formal arrows $\langle \mathscr{C}^{[n]}, \mathscr{W}^{[1]}_{\mathrm{reversed}} \rangle$ has elements
$$f = (f_1, \dots, f_n)\colon \mathrm{dom}f_1 \to \mathrm{cod}f_n$$
of formal strings of composable arrows, where for all i we either have $f_i \in \mathscr{C}^{[1]}$ or f_i is a formal reversal of an arrow in $\mathscr{W}$. Here $\mathrm{dom}f_1$ and $\mathrm{cod}f_n$ are defined to be the formal domain and codomain of the formal arrow f.
- The *free category* $\mathfrak{free}(\mathscr{C}, \mathscr{W}^{-1})$ has the same objects as $\mathscr{C}$ with classes of morphisms

$$\mathfrak{free}(\mathscr{C}, \mathscr{W}^{-1})(c, c') := \left\{ f \;\middle|\; n \in \mathbb{N},\ f \in \langle \mathscr{C}^{[n]},\ \mathscr{W}^{[1]}_{\text{reversed}} \rangle,\ \text{dom} f = c,\ \text{cod} f = c' \right\}$$

Empty strings $\mathscr{C}^{[0]} \ni \varnothing_c \colon c \to c$ at a particular object $c \in \mathscr{C}$, are interpreted as the identity at that object. Composition is defined via concatenation of strings.

- The class of morphisms $\text{Mor}(\mathfrak{free}(\mathscr{C}, \mathscr{W}^{-1}))$ gives rise to an equivalence relation $\sim$: An identity morphism 1_c in $\mathscr{C}$ for an object $c \in \mathscr{C}$ is identified with the empty string $\varnothing_c$, while any composable pair $(f, f') \in \mathscr{C}^{[2]}$ is identified with its composite $f' \circ f$. Moreover, for any $f \in \mathscr{W}$ we want the formal strings (f, f^{-1}) and (f^{-1}, f) to be equivalent to $1_{\text{dom} f}$ and $1_{\text{cod} f}$, respectively.
- The *localization* of $\mathscr{C}$ at $\mathscr{W}$ is the quotient category $\mathscr{C}[\mathscr{W}^{-1}]$ of $\mathfrak{free}(\mathscr{C}, \mathscr{W}^{-1})$ obtained from the equivalence relation $\sim$:

$$\mathscr{C}[\mathscr{W}^{-1}] := \frac{\mathfrak{free}(\mathscr{C}, \mathscr{W}^{-1})}{\sim}$$

The associated *localization functor* is the canonical functor

$$\gamma \colon \mathscr{C} \twoheadrightarrow \mathscr{C}[\mathscr{W}^{-1}]$$

- If $\mathscr{C}$ is a model category and $\mathscr{W}$ is the subcategory of weak equivalences, then the localization of $\mathscr{C}$ at $\mathscr{W}$ is called the *homotopy category* of $\mathscr{C}$ and is denoted by $\text{Ho}\mathscr{C}$.

Remark 5.19. Any category $\mathscr{C}$ is a category with weak equivalences by defining the corresponding wide subcategory $\mathscr{W}$ to be the category having the same objects as $\mathscr{C}$ with only isomorphisms as morphisms. The associated localization is then trivial:

$$\mathscr{C} \cong \mathscr{C}[\mathscr{W}^{-1}]$$

Notation 5.20. We will sometimes write $\gamma_{\mathscr{C}} \colon \mathscr{C} \twoheadrightarrow \mathscr{C}[\mathscr{W}^{-1}]$ for the corresponding localization functor, if ambiguity might arise otherwise.

Remark 5.21. It is clear from the definition that $\text{Ho}(\mathscr{C}^{\text{op}}) = (\text{Ho}\mathscr{C})^{\text{op}}$. Moreover, if $\mathscr{B}$ and $\mathscr{C}$ are both model categories, then $\text{Ho}(\mathscr{B} \times \mathscr{C})$ is isomorphic to $\text{Ho}\mathscr{B} \times \text{Ho}\mathscr{C}$.

Without passing to a higher set theoretic universe, $\text{Ho}\mathscr{C}$ might very well not be a category. It turns out however, as we will see soon enough (in Remark 5.31), that if $\mathscr{C}$ is a model category the localization of $\mathscr{C}$ at the weak equivalences will always turn out to be a well-defined category.

Lemma 5.22. *Let $\mathscr{C}$ be a category with weak equivalences $\mathscr{W}$.*

- *The pair $(\mathscr{C}[\mathscr{W}^{-1}],\ \gamma)$ enjoys a universal property: If a functor $\mathfrak{F} \colon \mathscr{C} \to \mathscr{D}$ sends morphisms of $\mathscr{W}$ to isomorphisms in $\mathscr{D}$, then there is a unique functor $\mathscr{C}[\mathscr{W}^{-1}] \to \mathscr{D}$ such that*

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\mathfrak{F}} & \mathscr{D} \\ {\scriptstyle \gamma}\big\downarrow & \nearrow_{\exists !} & \\ \mathscr{C}[\mathscr{W}^{-1}] & & \end{array}$$

commutes.

- *If $\mathscr{D}^{\mathscr{C}}_{\mathscr{W}} \subset \mathscr{D}^{\mathscr{C}}$ denotes the full subcategory of functors $\mathscr{C} \to \mathscr{D}$ which send morphisms of $\mathscr{W}$ to isomorphisms in $\mathscr{D}$, then precomposition with γ yields an isomorphism with inverse* loc*:*

$$\mathscr{D}^{\mathscr{C}[\mathscr{W}^{-1}]} \underset{\text{loc}}{\overset{\gamma^\star}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} \mathscr{D}^{\mathscr{C}}_{\mathscr{W}}$$

Remark 5.23. As is usual, if some mathematical object enjoys a universal property, then that mathematical object must be unique up to unique isomorphism. Indeed, in the above case one proves that if $\delta\colon \mathscr{C} \to \mathscr{E}$ is a functor that takes maps of $\mathscr{W}$ to isomorphisms and enjoys the same universal property as γ, then there is a unique isomorphism $\mathscr{C}[\mathscr{W}^{-1}] \xrightarrow{\zeta} \mathscr{E}$ such that $\zeta\gamma = \delta$.

Proof of Lemma 5.22. If $\mathfrak{F} \in \mathscr{D}^{\mathscr{C}}_{\mathscr{W}}$, then define $\text{loc}(\mathfrak{F})$ to be identical to $\mathfrak{F}$ on objects and morphisms of $\mathscr{C}$. For (formally) reversed arrows f^{-1} with $f \in \mathscr{W}$, define $\text{loc}(\mathfrak{F})(f^{-1}) = (\mathfrak{F}f)^{-1}$. This is a well-defined functor $\mathscr{C}[\mathscr{W}^{-1}] \to \mathscr{D}$ and it is the unique functor such that the diagram

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\mathfrak{F}} & \mathscr{D} \\ {\scriptstyle\gamma}\downarrow & \nearrow{\scriptstyle \text{loc}(\mathfrak{F})} & \\ \mathscr{C}[\mathscr{W}^{-1}] & & \end{array}$$

commutes. The isomorphism

$$\mathscr{D}^{\mathscr{C}}_{\mathscr{W}} \xrightarrow{\cong} \mathscr{D}^{\mathscr{C}[\mathscr{W}^{-1}]}$$

is then given by taking functors $\mathfrak{F} \in \mathscr{D}^{\mathscr{C}}_{\mathscr{W}}$ to the associated localized functors $\text{loc}(\mathfrak{F})$, and natural transformations $\zeta\colon \mathfrak{F} \to \mathfrak{U}$ are mapped to natural transformations $\text{loc}(\zeta)\colon \text{loc}(\mathfrak{F}) \to \text{loc}(\mathfrak{U})$ with components $\text{loc}(\zeta)_c = \zeta_c$ for all objects c. The inverse of this functor takes a functor $\mathfrak{U}\colon \mathscr{C}[\mathscr{W}^{-1}] \to \mathscr{D}$ and maps it to $\mathfrak{U}\gamma$, and natural transformations ζ are mapped to the whiskering $\zeta\gamma$. □

Definition 5.24. Let $\mathscr{C}$ be a model category. Denote by $\mathscr{C}_c, \mathscr{C}_f, \mathscr{C}_{cf}$ the full subcategories of $\mathscr{C}$ which contain all cofibrant, fibrant, and bifibrant (objects that are both fibrant and cofibrant) objects of $\mathscr{C}$, respectively.

Proposition 5.25. *Let $\mathscr{C}$ be a model category. Then the inclusion functors induce equivalences of categories:*

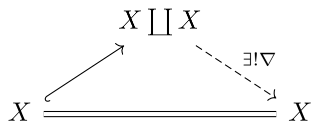
$$\begin{array}{ccccc} & & \text{Ho}\mathscr{C}_c & & \\ & \overset{\approx}{\nearrow} & & \overset{\approx}{\searrow} & \\ \text{Ho}\mathscr{C}_{cf} & & & & \text{Ho}\mathscr{C} \\ & \underset{\approx}{\searrow} & & \underset{\approx}{\nearrow} & \\ & & \text{Ho}\mathscr{C}_f & & \end{array}$$

Proof. We shall only verify the equivalence $\text{Ho}\mathscr{C}_c \to \text{Ho}\mathscr{C}$. Let $\mathscr{C}_c \overset{i}{\hookrightarrow} \mathscr{C}$ denote the inclusion functor. This functor preserves weak equivalences and thus induces a functor $\text{Ho}(i)\colon \text{Ho}\mathscr{C}_c \to \text{Ho}\mathscr{C}$ by Lemma 5.22. We then recall the cofibrant replacement functor $L\colon \mathscr{C} \to \mathscr{C}_c$ along with its associated natural trivial fibration $X \xrightarrow{l_X} LX$. Since the diagram

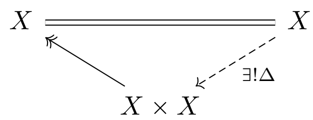
$$\begin{array}{ccc} LX & \xrightarrow{l_X} & X \\ {\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle f} \\ LY & \xrightarrow[l_Y]{} & Y \end{array}$$

commutes, the 2-out-of 3 property implies $Lf \in \mathscr{W}$ for any weak equivalence f in $\mathscr{C}$. Therefore, L preserves weak equivalences and again by Lemma 5.22 gives rise to a localized functor $\mathrm{Ho}(L)\colon \mathrm{Ho}\mathscr{C} \to \mathrm{Ho}\mathscr{C}_c$. The natural trivial fibration l is then understood as a natural weak equivalence $L \circ i \to 1_{\mathscr{C}_c}$ and $i \circ L \to 1_{\mathscr{C}}$. Hence this induces natural isomorphisms $\mathrm{Ho}(L \circ i) \to 1_{\mathrm{Ho}\mathscr{C}_c}$ and $\mathrm{Ho}(i \circ L) \to 1_{\mathrm{Ho}\mathscr{C}}$, as claimed. □

Now, if we are given a model category $\mathscr{C}$, then why do we call $\mathrm{Ho}\mathscr{C}$ the homotopy category of $\mathscr{C}$? What is so homotopical about it? There is a second way to construct $\mathrm{Ho}\mathscr{C}$ which we will sketch now. Before doing so, recall that if X is an object in $\mathscr{C}$ one defines the fold map $\nabla\colon X \coprod X \to X$ by means of the universal property of the coproduct:

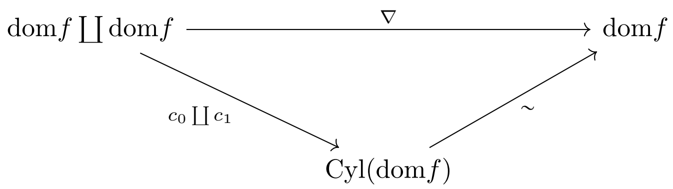


Analogously, the diagonal map $\Delta\colon X \to X \times X$ is defined by means of the universal property of the product:

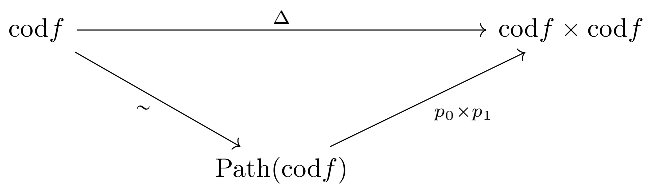


Definition 5.26. Let $\mathscr{C}$ be a model category, and fix morphisms f, g in $\mathscr{C}$ with the same domain and codomain.

- A *cylinder* for $\mathrm{dom} f$ is the data of a *cylinder object* $\mathrm{Cyl}(\mathrm{dom} f)$ for $\mathrm{dom} f$ along with a *factorization of the fold map*:

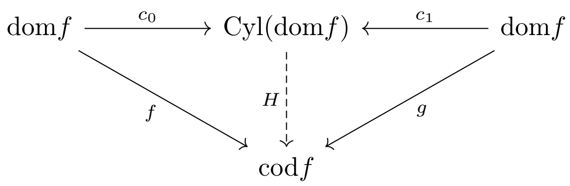


  such that $c_0 \coprod c_1$ is a cofibration and the map $\mathrm{Cyl}(\mathrm{dom} f) \xrightarrow{\sim} \mathrm{dom} f$ is a weak equivalence.
- A *path* for $\mathrm{cod} f$ is the data of a *path object* $\mathrm{Path}(\mathrm{cod} f)$ for $\mathrm{cod} f$ along with a *factorization of the diagonal map*:



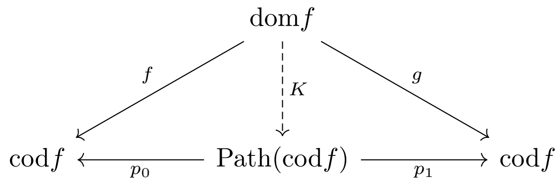
  such that $p_0 \times p_1$ is a fibration and the map $\mathrm{cod} f \xrightarrow{\sim} \mathrm{Path}(\mathrm{cod} f)$ is a weak equivalence.
- A *left homotopy* from f to g consists of a cylinder $\mathrm{Cyl}(\mathrm{dom} f)$ for $\mathrm{dom} f$ along with a morphism $H\colon \mathrm{Cyl}(\mathrm{dom} f) \to \mathrm{cod} f$ such that $Hc_0 = f$ and

$Hc_1 = g$. More concisely, we have a commutative diagram



In that case, we say that f and g are left homotopic and write $f \overset{l}{\sim} g$.

- A *right homotopy* from f to g consists of a path Path(cod$f$) for cod$f$ along with a morphism $K\colon \mathrm{dom}f \to \mathrm{Path}(\mathrm{cod}f)$ such that $p_0K = f$ and $p_1K = g$. More concisely, we have a commutative diagram

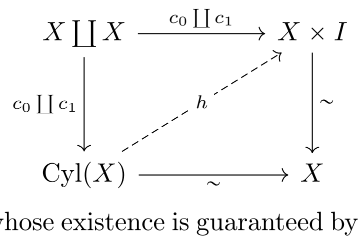


In that case, we say that f and g are right homotopic and write $f \overset{r}{\sim} g$.

- The morphisms f and g are said to be *homotopic*, if they are both left and right homotopic. In this case we write $f \sim g$.
- f is said to be a *homotopy equivalence* if there is a morphism $f'\colon \mathrm{cod}f \to \mathrm{dom}f$ such that $f'f \sim 1_{\mathrm{dom}f}$ and $ff' \sim 1_{\mathrm{cod}f}$.

Remark 5.27. A path object for X in $\mathscr{C}$ is the same as a cylinder object for X in the model category $\mathscr{C}^{\mathrm{op}}$. Similarily, the notions of right and left homotopy between two morphisms f and g are dual. Hence we may restrict ourselves to proving results about left homotopies and cylinder objects.

By means of our functorial factorizations we may construct a cylinder object $X \times I$ (the product here should be interpreted suggestively only) for any $X \in \mathscr{C}$. Indeed, apply the functorial factorization to the fold map $X \coprod X \to X$ to obtain a cofibration $X \coprod X \to X \times I$ along with a trivial fibration $X \times I \xrightarrow{\sim} X$. Dually, there is a path object X^I (this is yet again solely suggestive notation) for X by applying the other functorial factorization to the diagonal map, and in this case $X \xrightarrow{\sim} X^I$ is a trivial cofibration. In fact, if Cyl(X) is an arbitrary cylinder object for X, then there is a weak equivalence $\mathrm{Cyl}(X) \xrightarrow{\sim} B \times I$ compatible with the structure maps $c_0 \coprod c_1$ and the corresponding weak equivalences:



The morphism h is a lift whose existence is guaranteed by means of the LLP enjoyed by the fibration $c_0 \coprod c_1$ with respect to the trivial cofibration $X \times I \xrightarrow{\sim} X$. By the 2-out-of 3 property h is then a weak equivalence. Analogously, there is a weak equivalence $X^I \xrightarrow{\sim} \mathrm{Path}(X)$ for any path object Path(X) compatible with the associated structure maps.

Theorem 5.28 (Whitehead)**.** *Let $\mathscr{C}$ be a model category. A weak equivalence between two objects which are both fibrant and cofibrant is a homotopy equivalence.*

Proof. By means of the given functorial factorization in $\mathscr{C}$ and the 2-out-of 3 property any weak equivalence f in $\mathscr{C}$ factors through an object Z as a composition of a trivial cofibration followed by a trivial fibration. In particular, if $\mathrm{dom} f$ and $\mathrm{cod} f$ are both fibrant and cofibrant, so is Z. Hence it suffices to prove that trivial (co)fibrations between bifibrant objects are homotopy equivalences. So let f be a trivial fibration between bifibrant objects (the other case is dual). Then

$$\begin{array}{ccc} \varnothing & \longrightarrow & \mathrm{dom} f \\ {\scriptstyle \mathrm{Cof}\ni}\downarrow & \overset{f^{-1}}{\nearrow} & \downarrow{\scriptstyle f\in \mathrm{Fib}^{\simeq}} \\ \mathrm{cod} f & = & \mathrm{cod} f \end{array}$$

induces a right inverse f^{-1} for f. To see that f^{-1} is also a left inverse up to left homotopy, let $\mathrm{Cyl}(\mathrm{dom} f)$ be any cylinder object for $\mathrm{dom} f$, that is, a factorization

$$\begin{array}{ccc} \mathrm{dom} f \coprod \mathrm{dom} f & \xrightarrow{\nabla} & \mathrm{dom} f \\ {\scriptstyle \mathrm{Cof}\ni c_0 \coprod c_1}\downarrow & \nearrow{\scriptstyle \sigma\in \mathrm{Fib}^{\simeq}} & \\ \mathrm{Cyl}(\mathrm{dom} f) & & \end{array}$$

and consider the commuting square

$$\begin{array}{ccc} \mathrm{dom} f \coprod \mathrm{dom} f & \xrightarrow{f^{-1}f \coprod 1_{\mathrm{dom} f}} & \mathrm{dom} f \\ {\scriptstyle \mathrm{Cof}\ni c_0 \coprod c_1}\downarrow & \overset{\eta}{\nearrow} & \downarrow{\scriptstyle f\in \mathrm{Fib}^{\simeq}} \\ \mathrm{Cyl}(\mathrm{dom} f) & \xrightarrow[f\sigma]{} & \mathrm{cod} f \end{array}$$

which, by construction, admits a lift $\eta\colon \mathrm{Cyl}(\mathrm{dom} f) \to \mathrm{dom} f$ which then constitutes the wanted left homotopy. □

Proposition 5.29. *Let $\mathscr{C}$ be a model category. Then the following is true:*

- *For X' a cofibrant object and X a fibrant object of $\mathscr{C}$, the left and right homotopy relations coincide and are equivalence relations on $\mathscr{C}(X', X)$.*
- *The homotopy relation on the morphisms of $\mathscr{C}_{cf}$ is an equivalence relation and is compatible with composition.*
- *A map of $\mathscr{C}_{cf}$ is a weak equivalence if and only if it is a homotopy equivalence.*

Proof. See [19] Corollary 1.2.6 and 1.2.7 and Proposition 1.2.8. □

Corollary 5.30. *Let $\mathscr{C}$ be a model category and let $\frac{C_{cf}}{\sim}$ denote the quotient category obtained from $\mathscr{C}_{cf}$ by factoring out by the homotopy relation $\sim$. Let $\gamma\colon \mathscr{C}_{cf} \twoheadrightarrow \mathrm{Ho}\mathscr{C}_{cf}$ and $\delta\colon \mathscr{C}_{cf} \twoheadrightarrow \frac{\mathscr{C}_{cf}}{\sim}$ be the corresponding localization and quotient functors. Then there is a unique isomorphism of categories*

$$\frac{\mathscr{C}_{cf}}{\sim} \xrightarrow[\cong]{j} \mathrm{Ho}\mathscr{C}_{cf}$$

such that $j\delta = \gamma$. Furthermore, j is the identity on objects.

Proof. One shows that $\mathscr{C}_{cf}/_{\sim}$ satisfies the same universal property as $\mathrm{Ho}\mathscr{C}_{cf} \cong \mathrm{Ho}\mathscr{C}_{cf}$. For details see [19] corollary 1.2.9. □

Remark 5.31. By Corollary 5.30 and Propositon 5.25 we have

$$\frac{\mathscr{C}_{cf}}{\sim} \cong \mathrm{Ho}\mathscr{C}_{cf} \cong \mathrm{Ho}\mathscr{C}$$

which asserts that, whenever $\mathscr{C}$ is a model category, the localized category $\mathrm{Ho}\mathscr{C} = \mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}]$ is a well-defined category.

In [19] the next result is referred to as the fundamental theorem of model categories:

Theorem 5.32. *Let $\mathscr{C}$ be a model category. Denote by $\gamma\colon \mathscr{C} \twoheadrightarrow \mathrm{Ho}\mathscr{C}$ the canonical functor, and let L and R be cofibrant and fibrant replacement functors.*

- *There are natural isomorphisms*

$$\begin{array}{ccccc} & & \frac{\mathscr{C}(X,Y)}{\sim} & & \\ & & \Big\downarrow{\scriptstyle \cong}\ (\text{dotted}) & & \\ \frac{\mathscr{C}(LRX,LRY)}{\sim} & \xrightarrow{\ \cong\ } & \mathrm{Ho}\mathscr{C}(X,Y) & \xleftarrow{\ \cong\ } & \frac{\mathscr{C}(RLX,RLY)}{\sim} \\ & & \Big\uparrow{\scriptstyle \cong} & & \\ & & \frac{\mathscr{C}(LX,RY)}{\sim} & & \end{array}$$

  where the dotted arrow makes sense and is an isomorphism if and only if X is cofibrant and Y is fibrant.
- *The localization functor γ identifies left and right homotopic maps.*
- *Any morphism f in $\mathscr{C}$ such that γf is an isomorphism in $\mathrm{Ho}\mathscr{C}$ is a weak equivalence.*

Proof. See [19] Theorem 1.2.10. □

5.3. **Quillen Functors and Quillen Adjunctions.** We shall introduce the notions of Quillen functors and Quillen adjunctions in this chapter. The material is based on the corresponding chapters in [19].

As is quite clear from the definition, a model category is a category with extra homotopical structure. Standard adjunctions between model categories are not able to translate all the relevant structure from one model category to the other. The idea of Quillen adjunctions (and Quillen functors in general) is precisely this: To give us a nice enough notion of adjunctions between model categories which also preserve the given homotopical information.

Definition 5.33. Let $\mathscr{C}$ and $\mathscr{D}$ be model categories.

- A functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ is called a *left Quillen functor* if $\mathfrak{F}$ is a left adjoint and preserves cofibrations and trivial cofibrations.
- A functor $\mathfrak{U}\colon \mathscr{D} \to \mathscr{C}$ is called a *right Quillen functor* if $\mathfrak{U}$ is a right adjoint and preserves fibrations and trivial fibrations.
- An adjunction
$$\Big(\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}, \qquad \mathfrak{U}\colon \mathscr{D} \to \mathscr{C}, \qquad \varphi\colon \mathscr{D}(\mathfrak{F}, -) \xrightarrow{\cong} \mathscr{C}(-, \mathfrak{U})\Big)$$
  is called a *Quillen adjunction* if $\mathfrak{F}$ is a left Quillen functor.

Remark 5.34. Some immediate facts can be deduced:

- By Ken Brown's Lemma 5.17 every left Quillen functor preserves weak equivalences between cofibrant objects. Dually, every right Quillen functor preserves weak equivalences between fibrant objects. Thus a left Quillen functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ induces a functor $\mathrm{Ho}\mathfrak{F}\colon \mathrm{Ho}\mathscr{C}_c \to \mathrm{Ho}\mathscr{D}$. Analogously, a right Quillen functor $\mathfrak{U}\colon \mathscr{D} \to \mathscr{C}$ induces a functor $\mathrm{Ho}\mathfrak{U}\colon \mathrm{Ho}\mathscr{D}_f \to \mathrm{Ho}\mathscr{C}$.

- If we have a pair of Quillen adjunctions $(\mathfrak{F}, \mathfrak{U}, \varphi)\colon \mathscr{C} \to \mathscr{D}$ and $(\mathfrak{F}', \mathfrak{U}', \varphi')\colon \mathscr{D} \to \mathscr{E}$, then we can define their composition to be the adjunction
$$\Big(\mathfrak{F}'\mathfrak{F}\colon \mathscr{C} \to \mathscr{E}, \qquad \mathfrak{U}\mathfrak{U}'\colon \mathscr{E} \to \mathscr{C}, \qquad \varphi\varphi'\Big)$$
where $\varphi\varphi'$ is the composite natural isomorphism with components
$$\mathscr{E}(\mathfrak{F}'\mathfrak{F}c, e) \xrightarrow{\varphi'} \mathscr{D}(\mathfrak{F}c, \mathfrak{U}'e) \xrightarrow{\varphi} \mathscr{C}(c, \mathfrak{U}\mathfrak{U}'e)$$
Composition of (Quillen) adjunctions is associative and has units.
- If $(\mathfrak{F}, \mathfrak{U}, \varphi)\colon \mathscr{C} \to \mathscr{D}$ is a Quillen adjunction, then
$$(\mathfrak{F}, \mathfrak{U}, \varphi)^{\mathrm{op}} := (\mathfrak{U}, \mathfrak{F}, \varphi^{-1})\colon \mathscr{D}^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$$
is a Quillen adjunction.
- Let $(\mathfrak{F}, \mathfrak{U}, \varphi)$ be an adjunction $\mathscr{C} \to \mathscr{D}$ as above. The triple $(\mathfrak{F}, \mathfrak{U}, \varphi)$ is a Quillen adjunction if and only if $\mathfrak{U}$ is a right Quillen functor. Indeed, we have equivalent lifting problems:

$$\begin{array}{ccc} \mathfrak{F}(\mathrm{dom} f) & \longrightarrow & \mathrm{dom} g \\ {\scriptstyle \mathfrak{F}f}\downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle g} \\ \mathfrak{F}(\mathrm{cod} f) & \longrightarrow & \mathrm{cod} g \end{array} \qquad \Longleftrightarrow \qquad \begin{array}{ccc} \mathrm{dom} f & \longrightarrow & \mathfrak{U}(\mathrm{dom} g) \\ {\scriptstyle f}\downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle \mathfrak{U}g} \\ \mathrm{cod} f & \longrightarrow & \mathfrak{U}(\mathrm{cod} g) \end{array}$$

So for example, if $\mathfrak{F}f \in \mathrm{Cof}_{\mathscr{D}}$ for all $f \in \mathrm{Cof}_{\mathscr{C}}$, then $\mathfrak{F}f$ has the LLP with respect to all $g \in \mathrm{Fib}_{\mathscr{D}}^{\simeq}$. Thus by using the adjunction, $\mathfrak{U}g$ has the RLP with respect to all $f \in \mathrm{Cof}_{\mathscr{C}}$, and hence $\mathfrak{U}g \in \mathrm{Fib}_{\mathscr{C}}^{\simeq}$ for all $g \in \mathrm{Fib}_{\mathscr{D}}^{\simeq}$, as wanted.

We then also have a good notion of equivalence of model categories:

Definition 5.35. A *Quillen equivalence* of two model categories $\mathscr{C}$ and $\mathscr{D}$ is a Quillen adjunction
$$\mathscr{C} \underset{\mathfrak{U}}{\overset{\mathfrak{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\scriptstyle \perp \mathrm{Quillen}}}}} \mathscr{D}$$
such that for every cofibrant object $c \in \mathscr{C}$ and every fibrant object $d \in \mathscr{D}$, a morphism $c \to \mathfrak{U}d$ is a weak equivalence in $\mathscr{C}$ if and only if $\mathfrak{F}c \to d$ is a weak equivalence in $\mathscr{D}$.

5.4. **Important Model Structures.**

5.4.1. *Quillen Model Structure.* There are several model structures on the category of simplicial sets. The standard one is usually referred to as the Quillen model structure.

Definition 5.36. The *classical model structure* or *Quillen model structure* on the category of simplicial sets has the following distinguished classes of morphisms:

- *Cofibrations* are given by all monomorphisms $f\colon X \rightarrowtail Y$, i.e., $f_n\colon X_n \to Y_n$ is an injection for all $n \in \mathbb{N}$.
- *Weak equivalences* are weak homotopy equivalences, i.e., morphisms whose geometric realization is a weak homotopy equivalence of topological spaces.

- *Fibrations* are given by *Kan fibrations*, i.e., maps $f\colon X \to Y$ which have the RLP with respect to all horn inclusions

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \Delta^n & \longrightarrow & Y \end{array}$$

  for all $1 \leqslant k \leqslant n$.

In order to accentuate that we consider sSet endowed with the Quillen model structure, we shall sometimes write $\mathrm{sSet}_{\mathrm{Quillen}}$.

Remark 5.37. Let us deduce some consequences of the previous definition:

- Fibrant objects in the Quillen model structure are exactly Kan complexes:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \longrightarrow & \Delta^0 \end{array}$$

- A morphism $f\colon X \to Y$ of fibrant simplicial sets i.e. Kan complexes is a weak equivalence if and only if it induces isomorphisms on all simplicial homotopy groups.
- All simplicial sets are cofibrant with respect to the Quillen model structure.

5.4.2. *Classical Model Structure on* Top.

Definition 5.38. The *classical model structure* or *Quillen model structure* on Top has the following distinguished classes of morphisms:

- *Weak equivalences* are the *weak homotopy equivalences.*
- *Fibrations* are constituted by Serre fibrations:

$$\mathrm{Fib} = \mathscr{J}_{\mathrm{Top}}^{\boxslash} := \left(\left\{D^n \overset{(\mathrm{id},0)}{\hookrightarrow} D^n \times I\right\}_{n\in\mathbb{N}}\right)^{\boxslash}$$

The following result holds:

Theorem 5.39. *The adjunction*

$$\mathrm{sSet}_{\mathrm{Quillen}} \underset{\Pi_{\leqslant\infty}}{\overset{|-|}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Top}_{\mathrm{Quillen}}$$

constitutes a Quillen equivalence.

This will become important once we talk about ∞-*groupoids.*

5.4.3. *Thomason Model Structure.* The following is based on the Nlab articles Thomason model structure and Geometric realization of categories.

The category of small categories Cat may also be endowed with a suitable model structure. For its construction one uses the fully faithful nerve embedding

$$\mathfrak{N}\colon \mathrm{Cat} \to \mathrm{sSet}$$

along with geometric realization

$$|-|\colon \mathrm{sSet} \to \mathrm{Top}$$

Definition 5.40. The composition of functors

$$|\mathfrak{N}(-)|\colon \mathrm{Cat} \to \mathrm{Top}, \qquad \mathscr{C} \mapsto |\mathfrak{N}\mathscr{C}|$$

is referred to as *geometric realization of categories.*

Remark 5.41. By the homotopy hypothesis geometric realization of simplicial sets constitutes a Quillen equivalence between the homotopy theory of simplicial sets and the homtopy theory of topological spaces. In particular, for a category $\mathscr{C}$ the simplicial set $\mathfrak{N}\mathscr{C}$ is a model for an ∞-groupoid (or Kan complex), since it doesn't make a difference if we take the geometric realization of $\mathfrak{N}\mathscr{C}$ or of any fibrant replacement thereof, since these will be homtopy equivalent, e.g.,

$$|\mathrm{Ex}^{\infty}(\mathfrak{N}\mathscr{C})| \simeq |\mathfrak{N}\mathscr{C}|$$

Recall the barycentric subdivision functor sd and its right adjoint Ex from Example 2.32.

Definition 5.42. The *Thomason model structure* on Cat, denoted $\mathrm{Cat}_{\mathrm{Thomason}}$, is given by:

- A functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ is a *Thomason fibration* if and only if $\mathrm{Ex}^2\mathfrak{N}(\mathfrak{F})\colon \mathrm{Ex}^2\mathfrak{N}(\mathscr{C}) \to \mathrm{Ex}^2\mathfrak{N}(\mathscr{D})$ is a trivial Kan fibration.
- A functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ is a *Thomason weak equivalence* if and only if $\mathfrak{N}\mathfrak{F}\colon \mathfrak{N}\mathscr{C} \to \mathfrak{N}\mathscr{C}$ is a weak equivalence in the Quillen model structure on sSet.

Proposition 5.43. *The Quillen model structure on* sSet *is Quillen equivalent to the Thomason model structure on* Cat *which is witnessed by the Quillen equivalence:*

$$\mathrm{sSet}_{\mathrm{Quillen}} \underset{\mathrm{Ex}^2\mathfrak{N}}{\overset{\mathfrak{h}\mathrm{sd}^2}{\rightleftarrows}} \mathrm{Cat}_{\mathrm{Thomason}} \quad (\perp)$$

Proposition 5.44. *For a category $\mathscr{C}$, let $\nabla\mathscr{C}$ be the poset category of 1-simplices in the nerve $\mathfrak{N}\mathscr{C}$ ordered by inclusion. Then we have*

$$|\mathfrak{N}(\nabla\mathscr{C})| \simeq |\mathfrak{N}\mathscr{C}|$$

Another important result is the following:

Theorem 5.45 (Quillen Theorem A)**.** *Let $\mathscr{C}, \mathscr{D}$ be categories and let $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ be a functor. If for all $d \in \mathscr{D}$ the geometric realization $|\mathfrak{N}(\mathfrak{F}/d)|$ of the comma category $\mathfrak{F}/d$ is contractible, then*

$$\mathfrak{N}\mathfrak{F}\colon \mathfrak{N}\mathscr{C} \to \mathfrak{N}\mathscr{D}$$

is a weak homotopy equivalence.

For even more results and references see the Nlab article Geometric realization of categories.

5.4.4. *Model Structures on Functors.* There are two canonical ideas to put model structures on $\mathscr{D}^{\mathscr{C}}$ for $\mathscr{D}$ a model category and $\mathscr{C}$ a small category.

Definition 5.46. Denote by $\mathscr{D}^{\mathscr{C}} = [\mathscr{C}, \mathscr{D}]$ the functor category of functors $\mathscr{C} \to \mathscr{D}$.

- The *projective weak equivalences* or *projective fibrations* are those natural transformations which are objectwise weak equivalences or fibrations in $\mathscr{D}$.
- The *injective weak equivalences* or *injective cofibrations* are those natural transformations which are objectwise weak equivalences or cofibrations in $\mathscr{D}$.
- This gives rise to two different model structures on $\mathscr{D}^{\mathscr{C}}$, if these exist, which we write as $\mathscr{D}^{\mathscr{C}}_{\mathrm{proj}}$ and $\mathscr{D}^{\mathscr{C}}_{\mathrm{inj}}$. The first of these is referred to as the *projective model structure* and the latter is called the *injective model structure*.

5.4.5. *Reedy Model Structure.* There is a special kind of category, referred to as *Reedy category*, which always ensures the existence of a model structure called the *Reedy model structure* on the functor category $\mathscr{C}^{\mathscr{R}}$ for $\mathscr{R}$ a Reedy category and $\mathscr{C}$ a model category.

Definition 5.47. A *Reedy category* is a category $\mathscr{R}$ with two wide subcategories $\mathscr{R}^+$ and $\mathscr{R}^-$ and a total ordering, defined by a degree function $\deg\colon \mathrm{Ob}\mathscr{R} \to \alpha$, where α is an ordinal number, such that

- Every non-identity arrow in $\mathscr{R}^+$ raises degree.
- Every non-identity arrow in $\mathscr{R}^-$ lowers degree.
- For all objects $f \in \mathscr{R}^{[1]}$ there exists a unique $f^+ \in (\mathscr{R}^+)^{[1]}$ and a unique $f^- \in (\mathscr{R}^-)^{[1]}$ such that $f = f^+ f^-$.

Example 5.48. The simplex category Δ constitutes a Reedy category:

- The degree function is given by
$$\deg\colon \mathrm{Ob}\Delta \to \mathbb{N}, \qquad [n] \mapsto n$$
- A morphism $[k] \to [n]$ is in Δ^+ if and only if it is an injection.
- A morphism $[k] \to [n]$ is in Δ^- if and only if it is a surjection.

By switching Δ^+ and Δ^-, we may also realize Δ^{op} as a Reedy category. In fact, switching $\mathscr{R}^+$ with $\mathscr{R}^-$ yields a Reedy category $\mathscr{R}^{\mathrm{op}}$ for any Reedy category $\mathscr{R}$.

Theorem 5.49. *If $\mathscr{R}$ is a Reedy category and $\mathscr{C}$ is a model category, then there is a canonical induced model structure on the functor category $\mathscr{C}^{\mathscr{R}}$, denoted $\mathscr{C}^{\mathscr{R}}_{\mathrm{Reedy}}$, in which the weak equivalences are the objectwise weak equivalences in $\mathscr{C}$.*

If the model category $\mathscr{C}$ is nice enough we have the following:

Theorem 5.50. *Let $\mathscr{C}$ be a combinatorial model category and let $\mathscr{R}$ be a Reedy category. Then identity functors induce Quillen equivalences*

$$\mathscr{C}^{\mathscr{R}}_{\mathrm{proj}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp\text{Quillen}}} \mathscr{C}^{\mathscr{R}}_{\mathrm{Reedy}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp\text{Quillen}}} \mathscr{C}^{\mathscr{R}}_{\mathrm{inj}}$$

For more details see the Nlab page Reedy model structure.

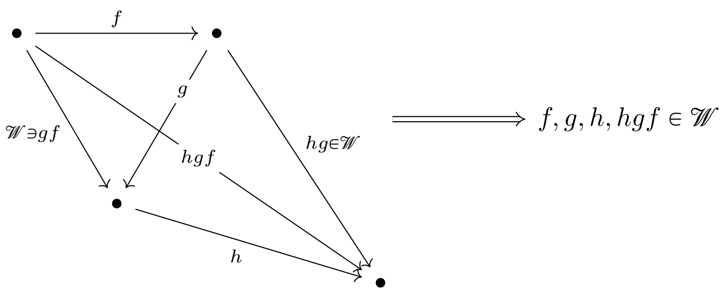
5.5. **Derived Functors.** This subsection is based on the corresponding chapters in [34] and [19].

We will define homotopy (co)limits as derived functors of a homotopy Kan extension that satisfy a universal property: the homotopy (co)limit functor is universal among homotopical approximations to the strict (co)limit functor. Let $\mathscr{C}$ be a category with weak equivalences and let $\mathscr{D}$ be a small diagram category. Then $\mathscr{C}^{\mathscr{D}}$ canonically turns into a category with weak equivalences by taking the weak equivalences to be those natural transformations which are objectwise weak equivalences in $\mathscr{C}$. Let $!\colon \mathscr{D} \to \star$ be the unique functor into the terminal category. Then we get the well known adjunction from Example 3.7:

$$\mathscr{C}^{\mathscr{D}} \;\underset{\lim=\mathrm{Ran}_!}{\overset{\mathrm{colim}=\mathrm{Lan}_!}{\rightleftarrows}}\; \mathscr{C}, \qquad \mathrm{colim} \dashv \mathrm{const} \dashv \lim$$

We will see that the globally defined homotopy limit and colimit are accordingly the left and right homotopy Kan extension along $!\colon \mathscr{D} \to \star$.

Definition 5.51. A *homotopical category* is a category $\mathscr{C}$ equipped with a *wide subcategory* $\mathscr{W}$ ($\mathscr{W}$ has the same objects as $\mathscr{C}$) such that $\mathscr{W}$ satisfies the *2-out-of-6* *property*: For any composable triple of arrows h, g, f if hg and gf are in $\mathscr{W}$, then so are f, g, h, and hgf. More diagrammatically,

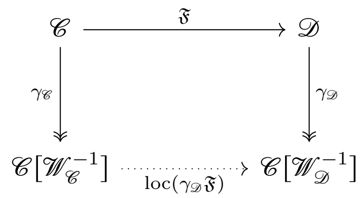


Remark 5.52. It is noteworthy that the 2-out-of 6 property is stronger than (and therefore implies) the 2-out-of 3 property. Still, it is shown, see [34] Remark 2.1.9, that the weak equivalences of any model category satisfy the 2-out-of 6 property. Thus any model category has an underlying homotopical category.

Example 5.53. Any category may be considered as a homotopical category by simply regarding the underlying *minimal homotopical category* taking the weak equivalences to be the isomorphisms in our category. Indeed, the class of isomorphisms satisfies the 2-out-of 6 property: For composable h, g, f such that gf and hg are isomorphisms the map g has a left inverse $f(gf)^{-1}$. As hg is an isomorphism g is monic and thus $f(gf)^{-1}$ must also be the right inverse of g. Thus g is an isomorphism, which already implies that f, h and hgf must also be isomorphisms.

Definition 5.54. A functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ between homotopical categories is said to be *homotopical* if it preserves weak equivalences, that is, $\mathfrak{F}(\mathscr{W}_{\mathscr{C}}) \subset \mathscr{W}_{\mathscr{D}}$.

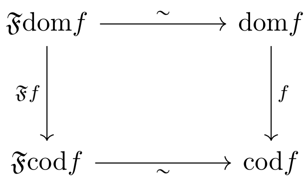
Remark 5.55. By the universal property of the corresponding localizations, a homotopical functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ induces a unique functor



commuting with the localizations.

Remark 5.56. Let $\mathscr{C}$ and $\mathscr{D}$ be homotopical categories. The universal property of the localization functor $\gamma_{\mathscr{C}}\colon \mathscr{C} \twoheadrightarrow \mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}]$ is 2-categorical: A natural transformation $\zeta\colon \mathfrak{F} \to \mathfrak{F}'$ between homotopical functors $\mathscr{C} \to \mathscr{D}$ descends to a unique natural transformation $\mathrm{loc}(\gamma_{\mathscr{D}}\zeta)\colon \mathrm{loc}(\gamma_{\mathscr{D}}\mathfrak{F}) \to \mathrm{loc}(\gamma_{\mathscr{D}}\mathfrak{F}')$. Conversely, a natural transformation between functors $\mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}] \to \mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}]$ descends to a natural transformation between functors $\mathscr{C} \to \mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}]$, but it might not be possible to lift this natural transformation along $\gamma_{\mathscr{D}}\colon \mathscr{D} \twoheadrightarrow \mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}]$.

Example 5.57. Any functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{C}$ equipped with a natural weak equivalence to or from the identity functor is homotopical: Indeed, if $\mathfrak{F} \xrightarrow{\sim} 1_{\mathscr{C}}$ is a natural weak equivalence, then any weak equivalence f is mapped to a weak equivalence $\mathfrak{F}f$ by the 2-out-of 3 property:

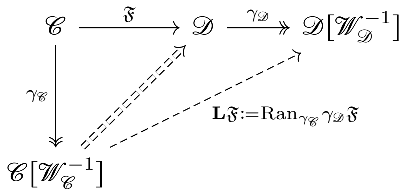


As a specific example, consider the functor that maps a space X to the cylinder $X \times I$, where $I = [0, 1]$. This functor is homotopical, because the canonical projections $X \times I \xrightarrow{\sim} X$ are natural weak equivalences.

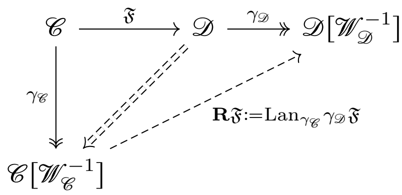
Many interesting functors between homotopical categories are not themselves homotopical however. What are we to do in such situations? Derived functors are defined to be the closest homotopical approximations of a (non-homotopical) functor between homotopical categories:

Definition 5.58. Let $\mathscr{C}$ and $\mathscr{D}$ be homotopical categories with wide subcategories $\mathscr{W}_{\mathscr{C}}$ and $\mathscr{W}_{\mathscr{D}}$, respectively, and let $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ be a functor. Denote by $\mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}]$ and $\mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}]$ the corresponding localizations (homotopy categories) as defined in Definition 5.18.

- The *total left derived functor* $\mathbf{L}\mathfrak{F}\colon \mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}] \to \mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}]$ is defined to be the right Kan extension of $\gamma_{\mathscr{D}}\mathfrak{F}$ along $\gamma_{\mathscr{C}}$:



- The *total right derived functor* $\mathbf{R}\mathfrak{F}\colon \mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}] \to \mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}]$ is defined to be the left Kan extension of $\gamma_{\mathscr{D}}\mathfrak{F}$ along $\gamma_{\mathscr{C}}$:



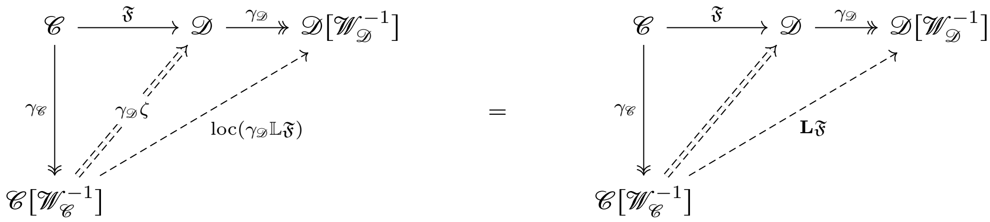
Remark 5.59. By the universal property of $\gamma_{\mathscr{C}}$, the functor $\mathbf{L}\mathfrak{F}$ may be considered as a homotopical functor $\mathbf{L}\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}]$.

Definition 5.60. Let $\mathscr{C}$ and $\mathscr{D}$ be homotopical categories, and consider a functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$.

- A *left derived functor* of $\mathfrak{F}$ is a homotopical functor $\mathbb{L}\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ equipped with a *comparison natural transformation* $\zeta\colon \mathbb{L}\mathfrak{F} \to \mathfrak{F}$ such that

$$\Big(\mathrm{loc}(\gamma_{\mathscr{D}}\mathbb{L}\mathfrak{F})\colon \mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}] \to \mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}], \qquad \gamma_{\mathscr{D}}\zeta\colon \gamma_{\mathscr{D}}\mathbb{L}\mathfrak{F} \to \gamma_{\mathscr{D}}\mathfrak{F}\Big)$$

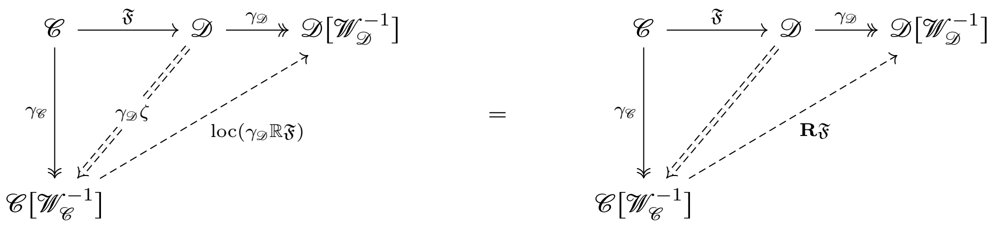
constitutes a total left derived functor of $\mathfrak{F}$:



- A *right derived functor* of $\mathfrak{F}$ is a homotopical functor $\mathbb{R}\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ equipped with a *comparison natural transformation* $\zeta\colon \mathfrak{F} \to \mathbb{R}\mathfrak{F}$ such that the data

$$\Big(\mathrm{loc}(\gamma_{\mathscr{D}}\mathbb{R}\mathfrak{F}), \qquad \gamma_{\mathscr{D}}\zeta\colon \gamma_{\mathscr{D}}\mathfrak{F} \to \gamma_{\mathscr{D}}\mathbb{R}\mathfrak{F}\Big)$$

  constitutes a total right derived functor of $\mathfrak{F}$:



Remark 5.61. The functor $\mathrm{loc}(\gamma_{\mathscr{D}}\mathbb{L}\mathfrak{F})$ in the above definition might seem confusing. However, after precomposing with $\gamma_{\mathscr{C}}$ we get

$$\mathrm{loc}(\gamma_{\mathscr{D}}\mathbb{L}\mathfrak{F})\gamma_{\mathscr{C}} = \gamma_{\mathscr{D}}\mathbb{L}\mathfrak{F}$$

yielding a natural transformation

$$\gamma_{\mathscr{D}}\zeta\colon \mathrm{loc}(\gamma_{\mathscr{D}}\mathbb{L}\mathfrak{F})\gamma_{\mathscr{C}} \to \gamma_{\mathscr{D}}\mathfrak{F}$$

Remark 5.62. After having had a look at the above definitions, we immediately realize that total left and right derived functors are unique up to unique isomorphisms (these are just Kan extensions after all). On the other hand, how about uniqueness for left and right derived functors? The expectation of course is that these are uniquely determined up to weak equivalence. This is indeed the case: If $\mathbb{L}\mathfrak{F}$ and $\widetilde{\mathbb{L}}\mathfrak{F}$ are two left derived functors for $\mathfrak{F}$, then

$$\mathrm{loc}(\gamma_{\mathscr{D}}\mathbb{L}\mathfrak{F}) \cong \mathrm{Ran}_{\gamma_{\mathscr{C}}}\gamma_{\mathscr{D}}\mathfrak{F} \cong \mathrm{loc}(\gamma_{\mathscr{D}}\widetilde{\mathbb{L}}\mathfrak{F})$$

and therefore we have a natural isomorphism $\gamma_{\mathscr{D}}\mathbb{L}\mathfrak{F} \cong \gamma_{\mathscr{D}}\widetilde{\mathbb{L}}\mathfrak{F}$, which makes precise what we mean by uniquely determined up to weak equivalence, since, if $\mathscr{C}$ is saturated (any model category is saturated), that is, any isomorphism in the localization of $\mathscr{D}$ is induced by a weak equivalence (this in particular concerns all the components of the above natural isomorphism), then the above natural isomorphism descends to a natural weak equivalence. There is a second question that one might want to ask. Namely, what if $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ is already homotopical. Then, since (total) left and right derived functors are thought to be the closest homotopical approximations to $\mathfrak{F}$, they should better agree with $\mathfrak{F}$. We postpone answering this question (see Corollary 5.67).

There is no guarantee for derived functors to always exist in general. However, there is a quite broad setting in which derived functors exist and admit simple constructions.

Definition 5.63. Let $\mathscr{C}$ and $\mathscr{D}$ be homotopical categories.

- A *left deformation* on $\mathscr{C}$ consists of a functor $L\colon \mathscr{C} \to \mathscr{C}$ together with a natural weak equivalence $l\colon L \xrightarrow{\sim} 1_{\mathscr{C}}$.
- A *right deformation* on $\mathscr{C}$ consists of a functor $R\colon \mathscr{C} \to \mathscr{C}$ together with a natural weak equivalence $r\colon 1_{\mathscr{C}} \xrightarrow{\sim} R$.

Remark 5.64. Let us point out some subtleties:

- If $\mathscr{C}$ admits a left deformation L, then L is homotopical. This follows from Example 5.57. If $\mathscr{C}_L$ is any full subcategory of $\mathscr{C}$ containing the image of Q, then the inclusion $\mathscr{C}_L \to \mathscr{C}$ and the left deformation $L\colon \mathscr{C} \to \mathscr{C}_L$ induce an equivalence of categories between $\mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}]$ and $\mathscr{C}_L[\mathscr{W}_{\mathscr{C}_L}^{-1}]$. This follows along the same lines as Proposition 5.25. Analogously, if $\mathscr{C}$ admits a right deformation R, then R is homotopical and any full subcategory $\mathscr{C}_R$ containing the image of R gives rise to an equivalence of categories $\mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}] \cong \mathscr{C}_R[\mathscr{W}_{\mathscr{C}_R}^{-1}]$
- The notion of left and right deformation are inspired by cofibrant and fibrant replacement functors $L, R\colon \mathscr{C} \to \mathscr{C}$ along with their natural weak equivalences $l\colon L \xrightarrow{\sim} 1_{\mathscr{C}}$ and $r\colon 1_{\mathscr{C}} \xrightarrow{\sim} R$ which always exist if $\mathscr{C}$ is a model category. Therefore, any model category gives rise to both a left and a right deformation.

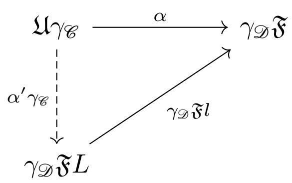
Definition 5.65. Let $\mathscr{C}$ be a homotopical category.

- A *left deformation* for a functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ between homotopical categories consists of a left deformation L for $\mathscr{C}$ such that $\mathfrak{F}$ is homotopical on an associated subcategory of cofibrant objects, i.e., $\mathfrak{F}$ is homotopical on $\mathscr{C}_L$ where $\mathscr{C}_L$ is any full subcategory containing the image of L.
- A *right deformation* for a functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ between homotopical categories consists of a right deformation R for $\mathscr{C}$ such that $\mathfrak{F}$ is homotopical on an associated subcategory of fibrant objects, i.e., $\mathfrak{F}$ is homotopical on $\mathscr{C}_R$ where $\mathscr{C}_R$ is any full subcategory containing the image of R.

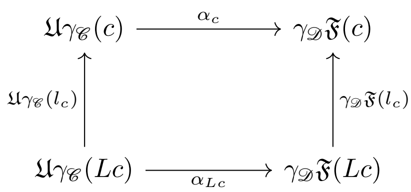
Theorem 5.66. *Let $\mathscr{C}$ and $\mathscr{D}$ be homotopical categories.*

- *If $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ has a left deformation $l\colon L \xrightarrow{\sim} 1_{\mathscr{C}}$, then $\mathbb{L}\mathfrak{F} = \mathfrak{F}L$ is a left derived functor of $\mathfrak{F}$ with comparison natural transformation $\mathfrak{F}l\colon \mathfrak{F}L \to \mathfrak{F}$.*
- *If $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ has a right deformation $r\colon 1_{\mathscr{C}} \xrightarrow{\sim} R$, then $\mathbb{R}\mathfrak{F} = \mathfrak{F}R$ is a right derived functor of $\mathfrak{F}$ with comparison natural transformation $\mathfrak{F}r\colon \mathfrak{F} \to \mathfrak{F}R$.*

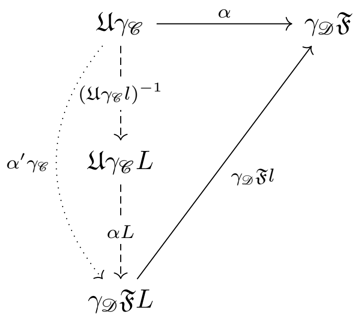
Proof. We have to show that for any functor $\mathfrak{U}\colon \mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}] \to \mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}]$ and any natural transformation $\alpha\colon \mathfrak{U}\gamma_{\mathscr{C}} \to \gamma_{\mathscr{D}}\mathfrak{F}$ there exists a unique morphism $\alpha'\colon \mathfrak{U} \to \operatorname{loc}(\gamma_{\mathscr{D}}\mathfrak{F}L)$ such that



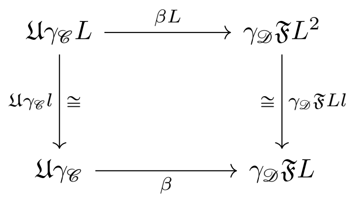
commutes. Existence of such a morphism α' is deduced as follows: The functor $\mathfrak{U}\gamma_{\mathscr{C}}$ is homotopical and therefore $\mathfrak{U}\gamma_{\mathscr{C}}l\colon \mathfrak{U}\gamma_{\mathscr{C}}Q \to \mathfrak{U}\gamma_{\mathscr{C}}$ is a natural isomorphism. Naturality of α implies commutativity of the diagram



which in turn yields that α factors through $\gamma_{\mathscr{D}}\mathfrak{F}L$ as



where α' is defined by $\mathrm{loc}(\alpha L(\mathfrak{U}\gamma_{\mathscr{C}}l)^{-1})$. Suppose now that we have yet another morphism $\beta\colon \mathfrak{U} \to \mathrm{loc}(\gamma_{\mathscr{D}}\mathfrak{F}L)$ such that $(\gamma_{\mathscr{D}}\mathfrak{F}l)\beta\gamma_{\mathscr{C}} = \alpha$. We first note that βL is uniquely determined: Since $\mathfrak{F}$ is homotopical on any full subcategory of cofibrant objects $\mathscr{C}_L$, $\mathfrak{F}lL$ is a natural weak equivalence. Thus $\gamma_{\mathscr{D}}\mathfrak{F}qQ$ is a natural isomorphism. Naturality implies commutativity of the diagram



which in turn yields uniqueness of β, since the vertical morphisms are isomorphisms.

□

Corollary 5.67. *If $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ is a homotopical functor between homotopical categories, then both the left and right derived functors $\mathbb{L}\mathfrak{F}$ and $\mathbb{R}\mathfrak{F}$ of $\mathfrak{F}$ exist and they both agree with $\mathfrak{F}$ up to weak equivalence.*

Proof. Since $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ is already homotopical, the identity functor $1_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$ is both a left as well as right deformation for $\mathfrak{F}$. Hence

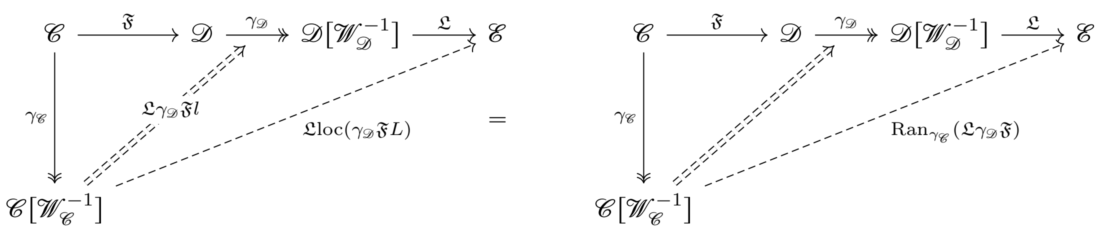
$$\mathbb{L}\mathfrak{F} \simeq \mathfrak{F} \simeq \mathbb{R}\mathfrak{F}$$

□

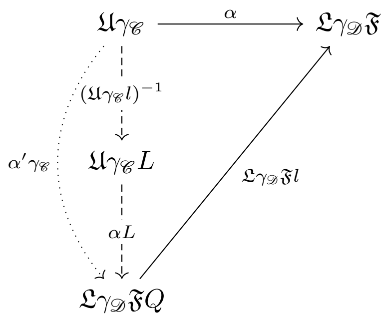
Lemma 5.68. *Let $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ be a functor between homotopical categories.*

- *If $\mathfrak{F}$ is left deformable, then its total left derived functor is an absolute left Kan extension.*
- *If $\mathfrak{F}$ is right deformable, then its total right derived functor is an absolute right Kan extension.*

Proof. The argument is along the same lines as in Theorem 5.66: Let $\mathfrak{L}\colon \mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}] \to \mathscr{E}$ be any functor. We have to show that



So let $\mathfrak{U}\colon \mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}] \to \mathscr{E}$ be a functor along with a natural transformation $\alpha\colon \mathfrak{U}\gamma_{\mathscr{C}} \to \mathfrak{L}\gamma_{\mathscr{D}}\mathfrak{F}$. The map $\mathfrak{U}\gamma_{\mathscr{C}} l$ is a natural isomorphism and hence by naturality of α we observe that α factors through $\mathfrak{L}\gamma_{\mathscr{D}}\mathfrak{F}L$ as



Showing uniqueness of α' is analogous as in the proof of Theorem 5.66. $\square$

Lemma 5.69. *Consider a pair of functors*

$$\mathscr{D} \xleftarrow{\ \mathfrak{U}\ } \mathscr{C} \xrightarrow{\ \mathfrak{F}\ } \mathscr{E}$$

- *If the right Kan extension* $\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}$ *of* $\mathfrak{F}$ *along* $\mathfrak{U}$ *exists and is absolute, then*
$$\mathscr{E}(\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}, e)\colon \mathscr{D}^{\mathrm{op}} \to \mathrm{Set}$$
is an absolute left Kan extension of $\mathscr{E}(\mathfrak{F}, e)$ *along* $\mathfrak{U}$.
- *If the left Kan extension* $\mathrm{Lan}_{\mathfrak{U}}\mathfrak{F}$ *of* $\mathfrak{F}$ *along* $\mathfrak{U}$ *exists and is absolute, then*
$$\mathscr{E}(e, \mathrm{Lan}_{\mathfrak{U}}\mathfrak{F})\colon \mathscr{D} \to \mathrm{Set}$$
is an absolute right Kan extension of $\mathscr{E}(e, \mathfrak{F})$ *along* $\mathfrak{U}$.

Proof. Since $\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}$ is assumed to be absolute, the functor $\mathscr{D}(-, e)\colon \mathscr{D} \to \mathrm{Set}^{\mathrm{op}}$ sends $\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}$ to a right Kan extension in $\mathrm{Set}^{\mathrm{op}}$. Hence, by duality, $\mathrm{Ran}_{\mathfrak{U}}\mathfrak{F}$ is taken to a left Kan extension of $\mathscr{E}(\mathfrak{F}, e)$ along $\mathfrak{U}$ in Set. $\square$

The proof of the following result is based on [20] (for an alternative proof which does not use the (co)end calculus, see [27]):

Theorem 5.70. *Let* $\mathscr{C}$ *and* $\mathscr{D}$ *be homotopical categories and let*

$$\mathscr{C} \underset{\mathfrak{U}}{\overset{\mathfrak{F}}{\rightleftarrows}} \mathscr{D} \quad (\mathfrak{F} \dashv \mathfrak{U})$$

be a pair of adjoint functors. If $\mathfrak{F}$ *admits an absolute total left derived functor* $\mathbf{L}\mathfrak{F}$ *and* $\mathfrak{U}$ *admits an absolute total right derived functor* $\mathbf{R}\mathfrak{U}$*, then the total derived functors form an adjunction between the corresponding localized categories:*

$$\mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}] \underset{\mathbf{R}\mathfrak{U}}{\overset{\mathbf{L}\mathfrak{F}}{\rightleftarrows}} \mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}] \quad (\mathbf{L}\mathfrak{F} \dashv \mathbf{R}\mathfrak{U})$$

Proof. Let us write $\mathscr{C}_{\sim} := \mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}]$ and $\mathscr{D}_{\sim} := \mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}]$. In the following calculation we make heavy use of Proposition 3.5, since all Kan extensions involved are absolute and therefore, in particular, pointwise:

$$\begin{aligned}
\mathscr{D}_{\sim}(\mathbf{L}\mathfrak{F}\gamma_{\mathscr{C}}c', \gamma_{\mathscr{D}}d') &\underset{\text{Lemma 5.69}}{\cong} \mathrm{Lan}_{\gamma_{\mathscr{C}}}(\mathscr{D}_{\sim}(\gamma_{\mathscr{D}}\mathfrak{F}, \gamma_{\mathscr{D}}d'))(\gamma_{\mathscr{C}}c') \\
&\cong \int^{c\in\mathscr{C}} \mathscr{C}_{\sim}(\gamma_{\mathscr{C}}c', \gamma_{\mathscr{C}}c) \times \mathscr{D}_{\sim}(\gamma_{\mathscr{D}}\mathfrak{F}c,\ \gamma_{\mathscr{D}}d') \\
&\cong \int^{c\in\mathscr{C}} \mathscr{C}_{\sim}(\gamma_{\mathscr{C}}c', \gamma_{\mathscr{C}}c) \times \Big(\int^{d\in\mathscr{D}} \mathscr{D}(\mathfrak{F}c, d) \times \mathscr{D}_{\sim}(\gamma_{\mathscr{D}}d, \gamma_{\mathscr{D}}d') \Big)
\end{aligned}$$

$$\cong \int^{c\in\mathscr{C}}\int^{d\in\mathscr{D}} \mathscr{C}_\sim(\gamma_\mathscr{C}c',\gamma_\mathscr{C}c)\times\mathscr{C}(c,\mathfrak{U}d)\times\mathscr{D}_\sim(\gamma_\mathscr{D}d,\gamma_\mathscr{D}d')$$

$$\cong \int^{d\in\mathscr{D}} \mathscr{C}_\sim(\gamma_\mathscr{C}c',\gamma_\mathscr{C}\mathfrak{U}d)\times\mathscr{D}_\sim(\gamma_\mathscr{D}d,\gamma_\mathscr{D}d')$$

$$\cong \mathrm{Lan}_{\gamma_\mathscr{D}}(\mathscr{C}_\sim(\gamma_\mathscr{C}c',\gamma_\mathscr{C}\mathfrak{U}))(\gamma_\mathscr{D}d')$$

$$\underset{\text{Lemma 5.69}}{\cong} \mathscr{C}_\sim(\gamma_\mathscr{C}c',\mathbf{R}\mathfrak{U}\gamma_\mathscr{D}d')$$

□

Corollary 5.71. *Let $\mathscr{C}$ and $\mathscr{D}$ be model categories. Then any Quillen adjunction*

$$\mathscr{C}\underset{\mathfrak{U}}{\overset{\mathfrak{F}}{\rightleftarrows}}\mathscr{D}\quad(\perp)$$

induces a derived adjunction

$$\mathrm{Ho}\mathscr{C}\underset{\mathbf{R}\mathfrak{U}}{\overset{\mathbf{L}\mathfrak{F}}{\rightleftarrows}}\mathrm{Ho}\mathscr{D}\quad(\perp)$$

Proof. The functors $\mathfrak{F}$ and $\mathfrak{U}$ are left resp. right deformable by Remark 5.64 and therefore admit total derived functors by Theorem 5.66. These total derived functors are absolute by Lemma 5.68. Theorem 5.70 then immediately implies the claim. □

Remark 5.72. In fact, it is not needed that the functors $\mathfrak{F}$ and $\mathfrak{U}$, in Corollary 5.71, form a Quillen adjunction. Theorem 5.70 implies the same claim by just demanding that $\mathfrak{F}$ is homotopical on the subcategory of cofibrant objects, while $\mathfrak{U}$ has to be homotopical on the subcategory of fibrant objects.

One then obtains a neat characterization of Quillen equivalences:

Proposition 5.73 ([19] Proposition 1.3.13)**.** *A Quillen adjunction $(\mathfrak{F},\mathfrak{U},\varphi)\colon\mathscr{C}\to\mathscr{D}$ is a Quillen equivalence if and only if The induced adjunction*

$$\mathrm{Ho}\mathscr{C}\underset{\mathbf{R}\mathfrak{U}}{\overset{\mathbf{L}\mathfrak{F}}{\rightleftarrows}}\mathrm{Ho}\mathscr{D}\quad(\perp)$$

is an equivalence of categories.

5.6. Model Categories with extra Structure.

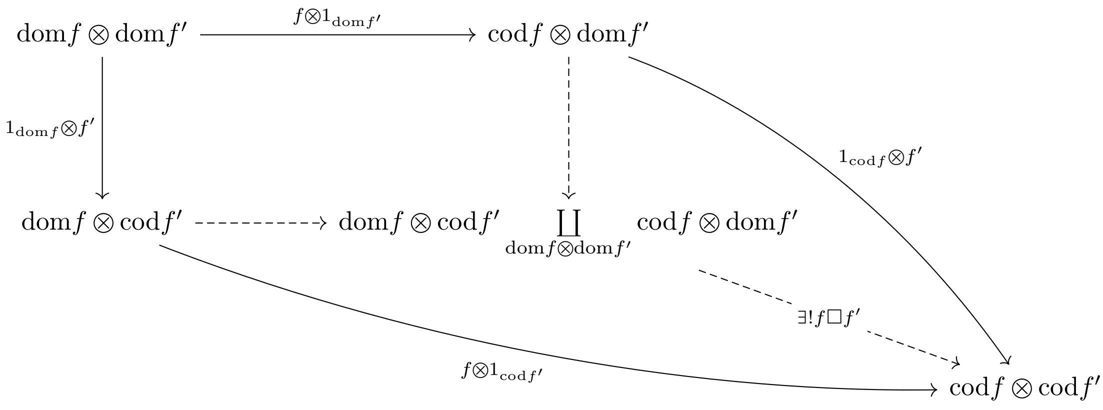
The following chapter is based on the Nlab-entry derived hom-functor.

We have seen what constitutes a model category and what it means for a category to be (symmetric) monoidal. Merging these two notions to obtain a concept of monoidal model category should result in a symbiosis of the monoidal structure with the model structure. Roughly put, a monoidal model category should be a model category which is also a closed monoidal category in a compatible way. Before getting to the precise definition let us briefly introduce a necessary preliminary notion:

Definition 5.74. Let $\otimes\colon\mathscr{E}_1\times\mathscr{E}_2\to\mathscr{E}_3$ be a functor and suppose $\mathscr{E}_3$ has pushouts. For morphisms f in $\mathscr{E}_1$ and f' in $\mathscr{E}_2$, the *pushout product* $f\Box f'$ is the morphism

$$(\mathrm{dom}f\otimes\mathrm{cod}f')\coprod_{\mathrm{dom}f\otimes\mathrm{dom}f'}(\mathrm{cod}f\otimes\mathrm{dom}f')\longrightarrow\mathrm{cod}f\otimes\mathrm{cod}f'$$

out of the pushout induced from the commuting diagram



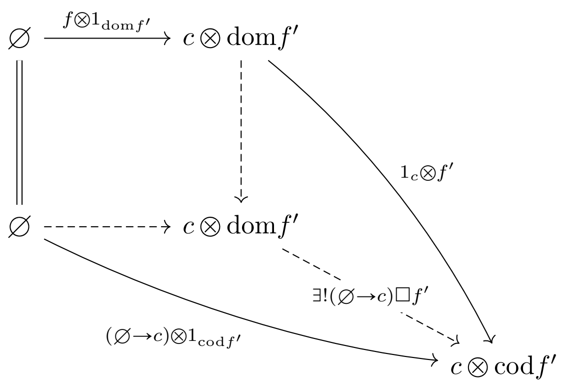
Since model categories are cocomplete (and complete), they allow for pushouts. Therefore, the pushout product for a pair of morphisms in $\mathscr{C}$ is well-defined.

Definition 5.75. A *symmetric monoidal model category* is a model category $\mathscr{C}$ equipped with a closed symmetric monoidal structure $(\mathscr{C}, \otimes, \mathbb{1}, \lambda, \rho)$ such that the following compatibility conditions are satisfied:

- *Pushout-product axiom:* For any pair of cofibrations f and f' in $\mathscr{C}$ their pushout-product $f \square f'$ with respect to the tensor functor $\otimes \colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is itself a cofibration, which, furthermore, is trivial if f or f' is trivial.
- *Unit axiom:* For every cofibrant object X and every cofibrant resolution $L\mathbb{1} \xrightarrow{l_{\mathbb{1}}} \mathbb{1}$ of the tensor unit $\mathbb{1}$, the resulting morphism
$$L\mathbb{1} \otimes X \xrightarrow{l_{\mathbb{1}} \otimes 1_X} \mathbb{1} \otimes X \xrightarrow{\cong} X$$
is a weak equivalence.

Remark 5.76. Some remarks are in order:

(i) For $c \in \mathscr{C}$ a cofibrant object, the pushout-product axiom implies that the functor $c \otimes - \colon \mathscr{C} \to \mathscr{C}$ preserves cofibrations and trivial cofibrations. Indeed, assume $\varnothing \to c$ to be a cofibration. Recall that the symmetric monoidal category $\mathscr{C}$ is closed, i.e., $- \otimes x$ has a right adjoint $[x, -]$ and thus preserves colimits for all objects $x \in \mathscr{C}$. Therefore, the diagram which induces the map $(\varnothing \to c) \square f'$ in the above definition of the pushout-product map boils down to



since $\varnothing \otimes \mathrm{dom} f' \cong \varnothing \cong \varnothing \otimes \mathrm{cod} f'$. Thus the (unique) induced morphism, which by assumption is a cofibration, amounts to $(\varnothing \to c) \square f' = 1_c \otimes f'$.

But this asserts that $c\otimes - : \mathscr{C} \to \mathscr{C}$ is a left Quillen functor. In particular, this tells us that the adjunction

$$\mathscr{C} \underset{[c,-]}{\overset{c\otimes -}{\rightleftarrows}} \mathscr{C} \quad (\bot)$$

is, in fact, a Quillen adjunction.

(*ii*) As a special case of the preceding remark, if $\mathbb{1}$ is cofibrant then the unit axiom is already implied by the pushout-product axiom: In fact, in this case $L\mathbb{1} \to \mathbb{1}$ is a weak equivalence between cofibrant objects and such morphisms are preserved by functors that preserve trivial cofibrations (this is Ken Brown's Lemma 5.17).

(*iii*) One can actually generalize the above definition further. This leads to the concept of a *left Quillen bifunctor*: Let $\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3$ be model categories. A functor $\otimes\colon \mathscr{E}_1 \times \mathscr{E}_2 \to \mathscr{E}_3$ is called a *left Quillen bifunctor* if
- it satisfies the pushout-product axiom.
- it preserves colimits separately in each variable.

Example 5.77. Consider the category of simplicial sets $\mathrm{sSet}_{\mathrm{Quillen}}$ endowed with the Quillen model structure. Fix cofibrations (monomorphisms in our case) f and f' in $\mathrm{sSet}_{\mathrm{Quillen}}$ and consider pushout square

$$\begin{array}{ccc}
\mathrm{dom}f \times \mathrm{dom}f' & \xrightarrow{f\times 1} & \mathrm{cod}f \times \mathrm{dom}f' \\
\downarrow{\scriptstyle 1\times f'} & & \downarrow \\
\mathrm{dom}f \times \mathrm{cod}f' & \dashrightarrow & (\mathrm{dom}f \times \mathrm{cod}f') \coprod_{\mathrm{dom}f\times\mathrm{dom}f'} (\mathrm{cod}f \times \mathrm{dom}f')
\end{array}$$

with maps $f\times 1\colon \mathrm{dom}f \times \mathrm{cod}f' \to \mathrm{cod}f \times \mathrm{cod}f'$, $1\times f'\colon \mathrm{cod}f \times \mathrm{dom}f' \to \mathrm{cod}f \times \mathrm{cod}f'$, and $f\square f' = (f\times 1)\coprod(1\times f')$ from the pushout to $\mathrm{cod}f \times \mathrm{cod}f'$.

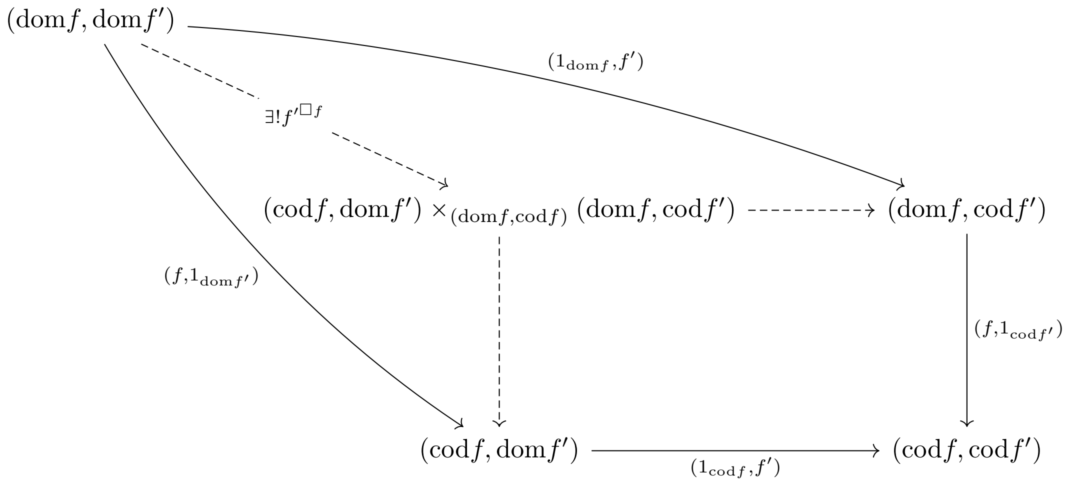
We note that the pushout-product may be explicitly computed as the map $(f \times 1)\coprod(1 \times f')$ and this map is certainly again a cofibration (monomorphism) if f and f' are cofibrations. For the case where f is a trivial cofibration and f' is a cofibration, see Proposition 4.2.8 in [19]. Similar arguments work to show that the injective model structure $(\mathrm{sSet}^{\mathscr{C}^{\mathrm{op}}}_{\mathrm{Quillen}})_{\mathrm{inj}}$ constitutes a monoidal model category.

We shall continue with yet another notion which builds on the previous ones. Before doing so however, let us discuss the dual notion of the pushout-product:

Definition 5.78. Let $(-,-)\colon \mathscr{E}_1 \times \mathscr{E}_2 \to \mathscr{E}_3$ be a functor and suppose $\mathscr{E}_3$ has pullbacks. For morphisms $f \in \mathscr{E}_1$ and $f' \in \mathscr{E}_2$, the *pullback-powering* $f'^{\square f}$ is the morphism

$$(\mathrm{dom}f, \mathrm{dom}f') \longrightarrow (\mathrm{cod}f, \mathrm{dom}f') \times_{(\mathrm{dom}f, \mathrm{cod}f)} (\mathrm{dom}f, \mathrm{cod}f')$$

into the pullback induced from the diagram



Definition 5.79. Let $\mathscr{V}$ be a monoidal model category. A *$\mathscr{V}$-enriched model category* is a $\mathscr{V}$-enriched category $\mathscr{C}$ which is both tensored and cotensored over $\mathscr{V}$ and which has the structure of a model category (the underlying category $\mathscr{C}_0$ is a model category) such that the following compatibility condition is satisfied:

- *Pullback-powering axiom:* For every cofibration $f \in \mathscr{C}$ and every fibration $f' \in \mathscr{C}$, the induced pullback-powering morphism $f'^{\square f}$ (with respect to the functor $\mathscr{C}(-,-)\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{V}$) is a fibration, which, furthermore, is trivial if f or f' is trivial.

Remark 5.80. The pullback-powering axiom, as in the above definition, is equivalent to the copower being a left Quillen bifunctor (it satisfies the pushout product axiom).

Any monoidal model category is in fact an enriched model category:

Proposition 5.81. *Any monoidal model category is an enriched model category over itself, via the enrichment of its underlying closed monoidal category.*

Proof. In order to prove this we shall make use of the Joyal-Tierney calculus. For this we shall use the notation

- $(-) \boxslash (-)$ for the lifting property,
- $(-)\square(-)$ for the pushout-product,
- $(-)^{\square(-)}$ for the pullback-powering.

We then have the following logical equivalences:

$$\mathrm{Cof}\square\mathrm{Cof} \subset \mathrm{Cof} \iff \mathrm{Cof}\square\mathrm{Cof} \boxslash \mathrm{Fib}^{\simeq} \iff \mathrm{Cof} \boxslash (\mathrm{Fib}^{\simeq})^{\square\mathrm{Cof}} \iff (\mathrm{Fib}^{\simeq})^{\square\mathrm{Cof}} \subset \mathrm{Fib}^{\simeq}$$

$$\mathrm{Cof}\square\mathrm{Cof}^{\simeq} \subset \mathrm{Cof}^{\simeq} \iff \mathrm{Cof}\square\mathrm{Cof}^{\simeq} \boxslash \mathrm{Fib} \iff \mathrm{Cof} \boxslash \mathrm{Fib}^{\square\mathrm{Cof}^{\simeq}} \iff \mathrm{Fib}^{\square\mathrm{Cof}^{\simeq}} \subset \mathrm{Fib}^{\simeq}$$

$$\mathrm{Cof}\square\mathrm{Cof}^{\simeq} \subset \mathrm{Cof}^{\simeq} \iff \mathrm{Cof}\square\mathrm{Cof}^{\simeq} \boxslash \mathrm{Fib} \iff \mathrm{Cof}^{\simeq} \boxslash \mathrm{Fib}^{\square\mathrm{Cof}} \iff \mathrm{Fib}^{\square\mathrm{Cof}} \subset \mathrm{Fib}$$

The statements on the far left constitute the pushout product axiom, while the statement on the far right yield the pullback-powering axiom. This shows equivalence of both these axioms and therefore the claim follows. For more details on the Joyal-Tierney calculus see the Nlab Enriched model category Example 4.1. and Joyal-Tierney calculus. □

Example 5.82. Proposition 5.81 leads to an onslaught of examples.

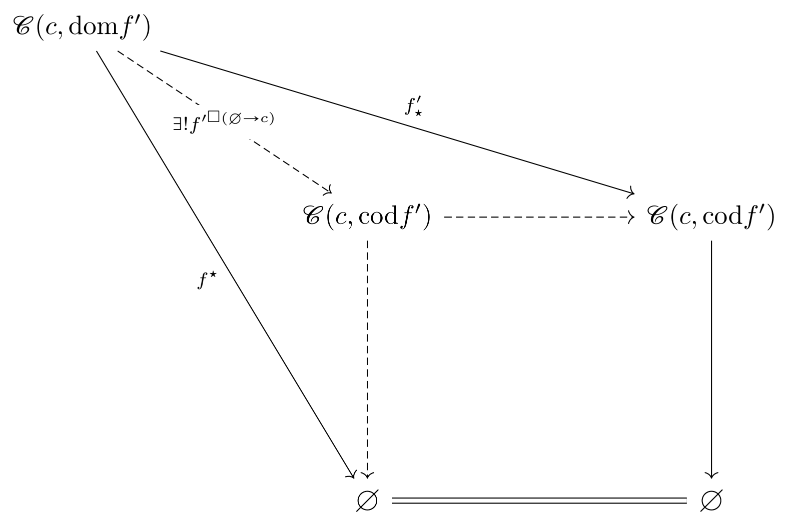
- The model category $\mathrm{sSet}_{\mathrm{Quillen}}$ is a monoidal model category and therefore, in particular, an enriched model category.

- The injective (or projective) model structure on simplicial presheaves, that is, $(\mathrm{sSet}^{\mathscr{C}^{\mathrm{op}}}_{\mathrm{Quillen}})_{\mathrm{inj}}$ (or $(\mathrm{sSet}^{\mathscr{C}^{\mathrm{op}}}_{\mathrm{Quillen}})_{\mathrm{proj}}$) is a monoidal model category, and therefore an enriched model category.

Lemma 5.83. *Let $\mathscr{C}$ be a $\mathscr{V}$-enriched model category.*

- *If $c \in \mathscr{C}$ is a cofibrant object, then the enriched hom-functor out of c*
$$\mathscr{C}(c,-)\colon \mathscr{C} \to \mathscr{V}$$
preserves fibrations and trivial fibrations.
- *If $\widetilde{c} \in \mathscr{C}$ is a fibrant object, then the enriched hom-functor into $\widetilde{c}$*
$$\mathscr{C}(-,\widetilde{c})\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{V}$$
sends cofibrations and trivial cofibrations in $\mathscr{C}$ to fibrations and trivial fibrations, respectively, in $\mathscr{V}$.

Proof. Let us suppose first that $\varnothing \to c$ is a cofibration. Since $\mathscr{C}$ is tensored and cotensored over $\mathscr{V}$ it follows that
$$\mathscr{C}(\varnothing,-) \cong \star, \qquad \mathscr{C}(-,\star) \cong \star$$
Indeed, for the first of these identities we calculate
$$\mathscr{C}(\varnothing,x) \cong \mathscr{C}(\varnothing \otimes \varnothing, x) \cong \mathscr{V}(\varnothing, \mathscr{C}(\varnothing,x)) \cong \star$$
where the first isomorphism in the above chain of morphisms follows from the fact that $\varnothing\otimes-$ preserves colimits (since it is a left adjoint), while the third isomorphism follows from the fact that $\mathscr{V}(-,\mathscr{C}(\varnothing,x))$ turns colimits to limits. The other identity follows analogously. Having gathered all that knowledge, the defining diagram for $f'^{\square(\varnothing\to c)}$ boils down to



But this means that the (trivial) fibration $f'^{\square(\varnothing\to c)}$ equals $f'_\star = \mathscr{C}(c,f)$, as wanted. □

Finally, we are ready to define a derived version of the enriched hom-functor: Let $L, R\colon \mathscr{C} \to \mathscr{C}$ be cofibrant and fibrant replacement functors along with the corresponding natural weak equivalences $l\colon L \xrightarrow{\sim} 1_{\mathscr{C}}$ and $r\colon 1_{\mathscr{C}} \xrightarrow{\sim} R$. The model category $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ then has a fibrant replacement functor
$$L^{\mathrm{op}} \times R\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to (\mathscr{C}^{\mathrm{op}} \times \mathscr{C})_f = (\mathscr{C}_c)^{\mathrm{op}} \times \mathscr{C}_f$$
along with a natural weak equivalence
$$l^{\mathrm{op}} \times r\colon L^{\mathrm{op}} \times R \xrightarrow{\sim} 1_{\mathscr{C}^{\mathrm{op}}\times\mathscr{C}}$$

Lemma 5.83 along with Ken Brown's Lemma 5.17 then shows that the enriched hom-bifunctor $\mathscr{C}(-,-)$ is homotopical if restricted to the full subcategory $(\mathscr{C}^{\mathrm{op}} \times \mathscr{C})_f$. Thus, $\mathscr{C}(-,-)$ admits a right deformation. Therefore the dual version of Theorem 5.66 may be applied to the enriched hom-bifunctor $\mathscr{C}(-,-)\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$, which guarantees the existence of the corresponding right derived functor:

Definition 5.84. Let $\mathscr{C}$ be an enriched model category. The enriched hom-functor $\mathscr{C}(-,-)\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ admits a right derived functor

$$\mathbb{R}\mathrm{Hom}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathscr{C}$$

referred to as the *(right) derived hom-functor*.

Remark 5.85. In the setting of cofibrant and fibrant replacement functors $L, R\colon \mathscr{C} \to \mathscr{C}$

$$\mathbb{R}\mathrm{Hom}(X, Y) \simeq \mathscr{C}(LX, RY)$$

for all objects $X, Y \in \mathscr{C}$.

5.6.1. *Simplicially Enriched Model Categories.*

Definition 5.86. A *cartesian closed monoidal category* is a cartesian closed category $\mathscr{C}$ equipped with a model structure such that the following axioms are satisfied:

- Pushout-product axiom,
- Pullback-powering axiom,
- Unit axiom.

Example 5.87. The category of simplicial sets endowed with the Quillen model structure $\mathrm{sSet}_{\mathrm{Quillen}}$ is a cartesian closed monoidal category.

Definition 5.88. A *simplicial model category* is an enriched model category where the enriching category is given by the cartesian closed model category $\mathrm{sSet}_{\mathrm{Quillen}}$ (the category of simplicial sets endowed with the Quillen model structure).

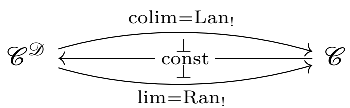
Example 5.89. Consider the model category $(\mathrm{sSet}^{\mathscr{C}^{\mathrm{op}}}_{\mathrm{Quillen}})_{\mathrm{inj}}$. The underlying category is sSet-enriched, that is, a simplicially enriched category. In fact, this even yields a simplicial model category.

Corollary 5.90. *Let $\mathscr{C}$ be a simplicial model category. If $c \in \mathscr{C}$ is cofibrant and $\tilde{c} \in \mathscr{C}$ is fibrant, then $\mathscr{C}(c, \tilde{c})$ is fibrant in $\mathrm{sSet}_{\mathrm{Quillen}}$, that is, $\mathscr{C}(c, \tilde{c})$ is a Kan complex.*

Proof. Follows immediately from Lemma 5.83. □

For more details and examples of simplicial model categories see the Nlab article Simplicial model category.

5.7. **Homotopy Limits and Colimits.** Let us get back to the adjunction which we had in the very beginning of 5.5:



If $\mathscr{C}$ is a model category (or any homotopical category really) and $\mathscr{D}$ is small, then view $\mathscr{C}^{\mathscr{D}}$ as a homotopical category with weak equivalences being those natural transformations which are objectwise weak equivalences in $\mathscr{C}$.

Definition 5.91. Let $\mathscr{C}$ be a model category and $\mathscr{D}$ be small.

- The *homotopy limit functor*, if it exists, is defined by the right derived functor of $\lim_{\mathscr{D}}$:

$$\operatorname{holim}_{\mathscr{D}} := \mathbb{R}\lim_{\mathscr{D}} \colon \mathscr{C}^{\mathscr{D}} \to \mathscr{C}$$

- The *homotopy colimit functor*, if it exists, is defined by the left derived functor of $\operatorname{colim}_{\mathscr{D}}$:

$$\operatorname{hocolim}_{\mathscr{D}} := \mathbb{L}\operatorname{colim}_{\mathscr{D}} \colon \mathscr{C}^{\mathscr{D}} \to \mathscr{C}$$

Remark 5.92. If both homotopy limit and colimit functors exist, then by Theorem 5.70 these give rise to adjunctions

$$\mathscr{C}^{\mathscr{D}}[\mathscr{W}_{\mathscr{C}^{\mathscr{D}}}^{-1}] \underset{\mathbf{R}\lim=\operatorname{loc}(\gamma_{\mathscr{C}}\mathbb{R}\lim)}{\overset{\mathbf{L}\mathrm{const}=\operatorname{loc}(\gamma_{\mathscr{C}^{\mathscr{D}}}\mathbb{L}\mathrm{const})}{\rightleftarrows}} \operatorname{Ho}\mathscr{C} \qquad \mathscr{C}^{\mathscr{D}}[\mathscr{W}_{\mathscr{C}^{\mathscr{D}}}^{-1}] \underset{\mathbf{R}\mathrm{const}=\operatorname{loc}(\gamma_{\mathscr{C}^{\mathscr{D}}}\mathbb{R}\mathrm{const})}{\overset{\mathbf{L}\operatorname{colim}=\operatorname{loc}(\gamma_{\mathscr{C}}\mathbb{L}\operatorname{colim})}{\rightleftarrows}} \operatorname{Ho}\mathscr{C}$$

Example 5.93. Let X_i be a collection of objects in a model category $\mathscr{C}$ indexed by some index set I. Their *homotopy product* is given as

$$\operatorname{holim}_{i\in I} X_i \simeq \prod_{i\in I} RX_i$$

for a fibrant replacement functor $R\colon \mathscr{C} \to \mathscr{C}$. Analogously, their *homotopy coproduct* is given by

$$\operatorname{hocolim}_{i\in I} X_i \simeq \coprod_{i\in I} LX_i$$

for a cofibrant replacement functor $L\colon \mathscr{C} \to \mathscr{C}$.

5.7.1. *Homotopy (Co)continuous functors.* Just like for ordinary limits and colimits, there should exist a notion of (homotopy) continuity and cocontinuity.

Definition 5.94. Consider the homotopical category $\mathscr{C}^{\mathscr{D}}$ (with objectwise weak equivalences) and suppose we are given a functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{C}'$, where $\mathscr{C}, \mathscr{C}'$ are homotopical categories.

- The functor $\mathfrak{F}$ is said to be *homotopy continuous*, if $\mathfrak{F}$ preserves homotopy limits, that is, for each $\mathfrak{U} \in \mathscr{C}^{\mathscr{D}}$ we have

$$\mathfrak{F}(\operatorname{holim}_{\mathscr{D}} \mathfrak{U}) \simeq \operatorname{holim}_{\mathscr{D}}(\mathfrak{F}\mathfrak{U})$$

- The functor $\mathfrak{F}$ is said to be *homotopy cocontinuous*, if $\mathfrak{F}$ preserves homotopy colimits, that is, for each $\mathfrak{U} \in \mathscr{C}^{\mathscr{D}}$ we have

$$\mathfrak{F}(\operatorname{hocolim}_{\mathscr{D}} \mathfrak{U}) \simeq \operatorname{hocolim}_{\mathscr{D}}(\mathfrak{F}\mathfrak{U})$$

Proposition 5.95 (Proposition 4.10 [1])**.** *Left derived functors of left Quillen functors preserve homotopy colimits and right derived functors of right Quillen functors preserve homotopy limits.*

Proof. Suppose we are given a Quillen adjunction

$$\mathscr{C} \underset{\mathfrak{U}}{\overset{\mathfrak{F}}{\rightleftarrows}} \mathscr{D}$$

then we get an induced Quillen adjunction

$$\mathscr{C}^{\mathscr{J}}_{\mathrm{inj}} \underset{\mathfrak{U}_\star}{\overset{\mathfrak{F}_\star}{\rightleftarrows}} \mathscr{D}^{\mathscr{J}}_{\mathrm{inj}}$$

We may therefore look at the commutative diagram of left Quillen functors

$$\begin{array}{ccc} \mathscr{C}^{\mathscr{J}}_{\mathrm{inj}} & \xrightarrow{\mathfrak{F}_\star} & \mathscr{D}^{\mathscr{J}}_{\mathrm{inj}} \\ \uparrow{\scriptstyle \mathrm{const}} & & \uparrow{\scriptstyle \mathrm{const}} \\ \mathscr{C} & \xrightarrow[\mathfrak{F}]{} & \mathscr{D} \end{array}$$

which induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Ho}\mathscr{C}^{\mathscr{J}}_{\mathrm{inj}} & \xrightarrow{\mathbf{L}\mathfrak{F}_\star} & \mathrm{Ho}\mathscr{D}^{\mathscr{J}}_{\mathrm{inj}} \\ \uparrow{\scriptstyle \mathbf{L}\mathrm{const}} & & \uparrow{\scriptstyle \mathbf{L}\mathrm{const}} \\ \mathrm{Ho}\mathscr{C} & \xrightarrow[\mathbf{L}\mathfrak{F}]{} & \mathrm{Ho}\mathscr{D} \end{array}$$

which yields commutativity of the diagram of right adjoints

$$\begin{array}{ccc} \mathrm{Ho}\mathscr{C}^{\mathscr{J}}_{\mathrm{inj}} & \xleftarrow{\mathbf{R}\mathfrak{U}_\star} & \mathrm{Ho}\mathscr{D}^{\mathscr{J}}_{\mathrm{inj}} \\ \downarrow{\scriptstyle \operatorname{holim}_{\mathscr{J}}} & & \downarrow{\scriptstyle \operatorname{holim}_{\mathscr{J}}} \\ \mathrm{Ho}\mathscr{C} & \xleftarrow[\mathbf{R}\mathfrak{U}]{} & \mathrm{Ho}\mathscr{D} \end{array}$$

which verifies that right derived functors of right Quillen functors indeed preserve homotopy limits. The other case is formally dual. □

5.8. **Homotopy (Co)Ends.** The following is based on [2].

Recall the adjunction from Remark 2.26 along with the dual statement about ends:

$$\mathscr{C}^{\mathscr{D}^{\mathrm{op}}\times\mathscr{D}} \underset{\prod_{\mathscr{D}(-,-)} := \mathscr{D}(-,-)\pitchfork -}{\overset{\int^{\mathscr{D}}}{\rightleftarrows}} \mathscr{C} \qquad \qquad \mathscr{C} \underset{\int_{\mathscr{D}}}{\overset{\coprod_{\mathscr{D}(-,-)} := \mathscr{D}(-,-)\odot -}{\rightleftarrows}} \mathscr{C}^{\mathscr{D}^{\mathrm{op}}\times\mathscr{D}}$$

By the end of this section we will have established the end as a right Quillen functor (nice pun, eh?) and thereby laying the groundwork for the notion of a homotopy end, i.e., the right derived functor of $\int_{\mathscr{D}}$. Let us start more generally and consider the following adjunction:

$$\mathscr{C}^{\mathscr{D}} \; \begin{array}{c} \xrightarrow{\mathfrak{U}_!} \\ \perp \\ \xleftarrow{\mathfrak{U}^\star} \\ \perp \\ \xrightarrow[\mathfrak{U}_\star]{} \end{array} \; \mathscr{C}^{\mathscr{D}'}$$

from Proposition 3.2, where we altered the notation considerably: $\mathfrak{U}_! := \mathrm{Lan}_{\mathfrak{U}}$ and $\mathfrak{U}_\star := \mathrm{Ran}_{\mathfrak{U}}$. Recall from chapter 5.4 that the functor category $\mathscr{C}^{\mathscr{D}}$, where $\mathscr{C}$ is a model category and $\mathscr{D}$ is any category, may give rise to two canonical model structures, which may or may not exist:

- The *projective model structure* $\mathscr{C}^{\mathscr{D}}_{\mathrm{proj}}$, where weak equivalences and fibrations are defined componentwise.
- The *injective model structure* $\mathscr{C}^{\mathscr{D}}_{\mathrm{inj}}$, where weak equivalences and cofibrations are defined componentwise.

Proposition 5.96. *Let $\mathscr{C}$ be a category, and $\mathfrak{U}\colon \mathscr{D} \to \mathscr{D}'$ be a functor. The adjunctions*

$$\mathscr{C}^{\mathscr{D}}_{\text{proj}} \underset{\mathfrak{U}^\star}{\overset{\mathfrak{U}_!}{\rightleftarrows}} \mathscr{C}^{\mathscr{D}'}_{\text{proj}} \qquad\qquad \mathscr{C}^{\mathscr{D}'}_{\text{inj}} \underset{\mathfrak{U}_\star}{\overset{\mathfrak{U}^\star}{\rightleftarrows}} \mathscr{C}^{\mathscr{D}}_{\text{inj}}$$

both constitute Quillen adjunctions, if the respective model structures exist.

Proof. We only need to verify that $\mathfrak{U}^\star$ both defines a left as well as right Quillen functor. But this is immediate, since, for example, if ψ is a fibration in $\mathscr{C}^{\mathscr{D}'}_{\text{proj}}$ then the resulting natural transformation $\mathfrak{U}^\star\psi = \psi\mathfrak{U}$ has only fibrations of $\mathscr{C}$ as components. Therefore, $\mathfrak{U}^\star\psi$ is a fibration. □

Definition 5.97. Let $\mathscr{C}$ be a model category and let $\mathscr{D}, \mathscr{D}'$ be categories.

- A *(trivial) simple projective cofibration* is a morphism in $\mathscr{C}^{\mathscr{D}}$ of the form
$$\coprod_{\mathscr{D}(d,-)} f\colon \coprod_{\mathscr{D}(d,-)} \operatorname{dom} f \to \coprod_{\mathscr{D}(d,-)} \operatorname{cod} f$$
for some (trivial) cofibration f in $\mathscr{C}$ and some object $d \in \mathscr{D}$.
- A *(trivial) simple injective fibration* is a morphism in $\mathscr{C}^{\mathscr{D}}$ of the form
$$\prod_{\mathscr{D}(-,d)} f\colon \prod_{\mathscr{D}(-,d)} \operatorname{dom} f \to \prod_{\mathscr{D}(-,d)} \operatorname{cod} f$$
for some (trivial) fibration f in $\mathscr{C}$ and some object $d \in \mathscr{D}$.

The names given in the above definitions are justified by the following:

Corollary 5.98. *Let $\mathscr{C}$ be a model category, and let $\mathfrak{U}\colon \mathscr{D} \to \mathscr{D}'$ be a functor.*

- *Any (trivial) simple projective cofibration is a (trivial) cofibration in $\mathscr{C}^{\mathscr{D}}_{\text{proj}}$.*
- *Any (trivial) simple injective fibration is a (trivial) fibration in $\mathscr{C}^{\mathscr{D}}_{\text{inj}}$.*
- *The left Kan extension $\mathfrak{U}_!\colon \mathscr{C}^{\mathscr{D}}_{\text{proj}} \to \mathscr{C}^{\mathscr{D}'}_{\text{proj}}$ preserves (trivial) simple projective cofibrations, that is,*
$$\mathfrak{U}_!\Big(\coprod_{\mathscr{D}(d,-)} f\Big) = \coprod_{\mathscr{D}'(\mathfrak{U}d,-)} f$$
- *The right Kan extension $\mathfrak{U}_\star\colon \mathscr{C}^{\mathscr{D}}_{\text{inj}} \to \mathscr{C}^{\mathscr{D}'}_{\text{inj}}$ preserves (trivial) simple injective fibrations, that is,*
$$\mathfrak{U}_\star\Big(\prod_{\mathscr{D}(-,d)} f\Big) = \prod_{\mathscr{D}(-,\mathfrak{U}d)} f$$

Proof. Fix some object $d \in \mathscr{D}$ and consider the inclusion functor $\{d\} \overset{\iota}{\hookrightarrow} \mathscr{D}$ of the subcategory $\{d\}$ with only one object. By Proposition 5.96 the left Kan extension $\iota_!\colon \mathscr{C}^{\{d\}}_{\text{proj}} \cong \mathscr{C} \to \mathscr{C}^{\mathscr{D}}_{\text{proj}}$ is left Quillen and therefore preserves (trivial cofibrations). Hence, for a (trivial) cofibration f in $\mathscr{C}$ the morphism

$$\iota_! f = \int^{\{d\}} \mathscr{D}(d,-) \odot f \cong \coprod_{\mathscr{D}(d,-)} f$$

is a (trivial) cofibration. Moreover, applying Kan extensions to the diagram

$$\begin{array}{ccc} \{d\} & \hookrightarrow & \mathscr{D} \\ \downarrow & & \downarrow \mathfrak{U} \\ \{\mathfrak{U}d\} & \hookrightarrow & \mathscr{D}' \end{array}$$

and using that Kan extensions respect compositions yields the remaining claim about simple projective cofibrations. The remaining claims follow by duality. □

Corollary 5.99. *Let* $\mathrm{const}\colon \mathscr{C} \to \mathscr{C}^{\mathscr{D}}$ *be the constant diagram functor.*

- *If* $\mathscr{C}^{\mathscr{D}}_{\mathrm{proj}}$ *exists, then*

$$\mathscr{C}^{\mathscr{D}}_{\mathrm{proj}} \underset{\mathrm{const}}{\overset{\mathrm{colim}}{\rightleftarrows}} \mathscr{C} \quad (\bot)$$

is a Quillen adjunction.

- *If* $\mathscr{C}^{\mathscr{D}}_{\mathrm{inj}}$ *exists, then*

$$\mathscr{C} \underset{\lim}{\overset{\mathrm{const}}{\rightleftarrows}} \mathscr{C}^{\mathscr{D}}_{\mathrm{inj}} \quad (\bot)$$

is a Quillen adjunction.

Proof. Apply Proposition 5.96 to the functor $\mathscr{D} \to \star$. □

Finally we are ready to prove the following:

Theorem 5.100. *Let* $\mathscr{C}$ *be a model category and* $\mathscr{D}$ *be a category. Regard* $\mathscr{C}^{\mathscr{D}^{\mathrm{op}}\times\mathscr{D}}$ *as a model category in any of the following ways (provided these model structures exist):*

- $\mathscr{C}^{\mathscr{D}^{\mathrm{op}}\times\mathscr{D}} = (\mathscr{C}^{\mathscr{D}^{\mathrm{op}}}_{\mathrm{proj}})^{\mathscr{D}}_{\mathrm{inj}}$
- $\mathscr{C}^{\mathscr{D}^{\mathrm{op}}\times\mathscr{D}} = (\mathscr{C}^{\mathscr{D}}_{\mathrm{proj}})^{\mathscr{D}^{\mathrm{op}}}_{\mathrm{inj}}$
- $\mathscr{C}^{\mathscr{D}^{\mathrm{op}}\times\mathscr{D}} = \mathscr{C}^{\mathscr{D}^{\mathrm{op}}\times\mathscr{D}}_{\mathrm{Reedy}}$ *if* $\mathscr{D}$ *is Reedy.*

Then the end functor

$$\int_{\mathscr{D}} \colon \mathscr{C}^{\mathscr{D}^{\mathrm{op}}\times\mathscr{D}} \to \mathscr{C}$$

is a right Quillen functor.

Proof. We shall only prove the result for the case $\mathscr{C}^{\mathscr{D}^{\mathrm{op}}\times\mathscr{D}} = (\mathscr{C}^{\mathscr{D}^{\mathrm{op}}}_{\mathrm{proj}})^{\mathscr{D}}_{\mathrm{inj}}$. The second case then follows by duality and the third case is shown in [2] Theorem 4.1. It suffices to check that the left adjoint

$$\coprod_{\mathscr{D}(-,-)} \dashv \int_{\mathscr{D}}$$

takes (trivial) cofibrations in $\mathscr{C}$ to (trivial) cofibrations in $(\mathscr{C}^{\mathscr{D}^{\mathrm{op}}}_{\mathrm{proj}})^{\mathscr{D}}_{\mathrm{inj}}$. If f is a (trivial) cofibration in $\mathscr{C}$, then the map

$$\coprod_{\mathscr{D}(-,-)} f \colon \coprod_{\mathscr{D}(-,-)} \mathrm{dom} f \to \coprod_{\mathscr{D}(-,-)} \mathrm{cod} f$$

must be a projective (trivial) cofibration objectwise. However, for a fixed object $d \in \mathscr{D}$, the map

$$\coprod_{\mathscr{D}(-,d)} f \colon \coprod_{\mathscr{D}(-,d)} \mathrm{dom} f \to \coprod_{\mathscr{D}(-,d)} \mathrm{cod} f$$

is a simple (trivial) projective cofibration in $\mathscr{C}^{\mathscr{D}^{\mathrm{op}}}$ and therefore the claim follows from Corollary 5.98. □

Recall the following definition:

Definition 5.101. Let $\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3$ be model categories.

- A functor $\otimes\colon \mathscr{E}_1 \times \mathscr{E}_2 \to \mathscr{E}_3$ is called a *left Quillen bifunctor* if
  - it satisfies the pushout-product axiom.
  - it preserves colimits separately in each variable.
- A functor $(-,-)\colon \mathscr{E}_1 \times \mathscr{E}_2 \to \mathscr{E}_3$ is called a *right Quillen bifunctor* if

– it satisfies the pullback-powering axiom.
– it preserves limits separately in each variable.

Lemma 5.102. *Let $\mathscr{E}_1, \mathscr{E}_2$ and $\mathscr{E}_3$ be model categories.*

- *Any left Quillen bifunctor $\otimes\colon \mathscr{E}_1 \times \mathscr{E}_2 \to \mathscr{E}_3$ gives rise to left Quillen functors $e_1 \otimes - \colon \mathscr{E}_2 \to \mathscr{E}_3$ and $- \otimes e_2 \colon \mathscr{E}_1 \to \mathscr{E}_3$ for all $e_1 \in \mathscr{E}_1$ and for all $e_2 \in \mathscr{E}_2$.*
- *Any right Quillen bifunctor $(-,-)\colon \mathscr{E}_1 \times \mathscr{E}_2 \to \mathscr{E}_3$ gives rise to right Quillen functors $(e_1, -)\colon \mathscr{E}_2 \to \mathscr{E}_3$ and $(-, e_2)\colon \mathscr{E}_1 \to \mathscr{E}_3$ for all $e_1 \in \mathscr{E}_1$ and for all $e_2 \in \mathscr{E}_2$.*

Proof. The same arguments as in Remark 5.76 work to show that any left Quillen bifunctor indeed gives rise to two functors $e_1 \otimes -$ and $- \otimes e_2$ which both preserve cofibrations and trivial cofibrations. These two functors then furthermore fit into an adjunction since the are both cocontinuous (by assumption). Indeed, by Theorem 3.10 it suffices to check that $\mathrm{Lan}_{e_1 \otimes -} 1$ and $\mathrm{Lan}_{- \otimes e_2} 1$ both exist and are preserved by $e_1 \otimes -$ and $- \otimes e_2$, respectively. Existence follows immediately from the fact that model categories are cocomplete by definition. Preservation is a quick coend calculation:

$$\begin{aligned} e_1 \otimes \mathrm{Lan}_{e_1 \otimes -} 1 &\cong e_1 \otimes \int^{e} \mathscr{E}_3(e_1 \otimes e, -) \odot e \\ &\cong \int^{e} e_1 \otimes (\mathscr{E}_3(e_1 \otimes e, -) \odot e) \\ &\cong \int^{e} \mathscr{E}_3(e_1 \otimes e, -) \odot (e_1 \otimes e) \\ &\cong \mathrm{Lan}_{e_1 \otimes -}(e_1 \otimes -) \end{aligned}$$

Analogously for $- \otimes e_2$. This proves that both $e_1 \otimes -$ and $- \otimes e_2$ are left Quillen functors. The claim regarding right Quillen bifunctors follows formally by duality. □

For $\mathscr{C}$ a combinatorial simplicial model category ($\mathscr{C}$ is nice enough) and $\mathscr{D}$ any simplicially enriched category the projective and injective model structures on $\mathscr{C}^{\mathscr{D}}$ both exist and each of these themselves come equipped with a combinatorial simplicial model structure. This ensures the existence of cofibrant replacement functors

$$L_{\mathrm{proj}} \colon \mathscr{C}^{\mathscr{D}}_{\mathrm{proj}} \to \mathscr{C}^{\mathscr{D}}_{\mathrm{proj}}, \qquad L_{\mathrm{inj}} \colon \mathscr{C}^{\mathscr{D}}_{\mathrm{inj}} \to \mathscr{C}^{\mathscr{D}}_{\mathrm{inj}}$$

That $\mathscr{C}$ is a simplicial model category moreover ensures that it is tensored over sSet

$$\odot \colon \mathscr{C} \times \mathrm{sSet} \to \mathscr{C}$$

and that the tensoring is a *left Quillen bifunctor.*

Proposition 5.103 (Remark A.2.9.27 in [25])**.** *Suppose that $\mathscr{C}$ and $\mathscr{D}$ are combinatorial model categories and let $\mathscr{J}$ be an arbitrary small category.*

- *Then any left Quillen bifunctor $\otimes\colon \mathscr{C} \times \mathscr{D} \to \mathscr{E}$ induces left Quillen bifunctors*

$$\int^{\mathscr{J}} - \otimes - \colon \mathscr{C}^{\mathscr{J}}_{\mathrm{proj}} \times \mathscr{D}^{\mathscr{J}^{\mathrm{op}}}_{\mathrm{inj}} \to \mathscr{E}, \qquad \int^{\mathscr{J}} - \otimes - \colon \mathscr{C}^{\mathscr{J}}_{\mathrm{inj}} \times \mathscr{D}^{\mathscr{J}^{\mathrm{op}}}_{\mathrm{proj}} \to \mathscr{E}$$

- *Then any right Quillen bifunctor* $\{-,-\}\colon \mathscr{C}\times\mathscr{D}\to\mathscr{E}$ *induces right Quillen bifunctors*
$$\int_{\mathscr{J}}\{-,-\}\colon \mathscr{C}^{\mathscr{J}}_{\mathrm{proj}}\times\mathscr{D}^{\mathscr{J}^{\mathrm{op}}}_{\mathrm{proj}}\to\mathscr{E},\qquad \int_{\mathscr{J}}\{-,-\}\colon \mathscr{C}^{\mathscr{J}}_{\mathrm{inj}}\times\mathscr{D}^{\mathscr{J}^{\mathrm{op}}}_{\mathrm{inj}}\to\mathscr{E}$$

Proof. The above functor is certainly cocontinuous in each of the two variables (by assumption). Therefore, it suffices to prove that any projective cofibration f in $\mathscr{C}^{\mathscr{J}}_{\mathrm{proj}}$ and any injective cofibration f' in $\mathscr{D}^{\mathscr{J}^{\mathrm{op}}}_{\mathrm{inj}}$, the induced map

$$\int^{\mathscr{J}}\mathrm{dom}\,f\otimes\mathrm{cod}\,f' \coprod_{\int^{\mathscr{J}}\mathrm{dom}f\otimes\mathrm{dom}f'} \int^{\mathscr{J}}\mathrm{cod}\,f\otimes\mathrm{dom}\,f' \xrightarrow{f\Box f'} \int^{\mathscr{J}}\mathrm{cod}\,f\otimes\mathrm{cod}\,f' \tag{3}$$

is a cofibration in $\mathscr{E}$ which is trivial if either f or f' is trivial. It suffices to check this (see [25]) that this holds for all simple projective cofibrations f of the form
$$\coprod_{\mathscr{J}(j,-)}c\longrightarrow\coprod_{\mathscr{J}(j,-)}c'$$
But then the arrow (3) boils down to
$$(c\otimes\mathrm{cod}\,f'(j))\coprod_{c\otimes\mathrm{dom}\,f(j)}(c'\otimes\mathrm{dom}\,f'(j))\longrightarrow c'\otimes\mathrm{cod}\,f'(j) \tag{4}$$
since, for example,
$$\begin{aligned}\mathscr{E}\Big(\int^{\tilde{j}\in\mathscr{J}}\coprod_{\mathscr{J}(j,\tilde{j})}f\otimes\mathrm{cod}\,f'(\tilde{j}),e\Big)&\cong\int_{\tilde{j}\in\mathscr{J}}\mathscr{E}\Big(\coprod_{\mathscr{J}(j,\tilde{j}}c\otimes\mathrm{cod}\,f'(\tilde{j}),e\Big)\\&\cong\int_{\tilde{j}\in\mathscr{J}}\mathrm{Set}(\mathscr{J}(j,\tilde{j}),\ \mathscr{E}(c\otimes\mathrm{cod}\,f'(\tilde{j}),e))\\&\cong\mathrm{Set}^{\mathscr{J}}(\mathscr{J}(j,-),\ \mathscr{E}(c\otimes\mathrm{cod}\,f',e))\\&\cong\mathscr{E}(c\otimes\mathrm{cod}\,f'(j),e)\end{aligned}$$
and therefore
$$\int^{\tilde{j}\in\mathscr{J}}\coprod_{\mathscr{J}(j,\tilde{j})}c\otimes\mathrm{cod}\,f'(\tilde{j})\cong c\otimes\mathrm{cod}\,f'(j)$$
and analogously for the other components in equation (3). But then (4) readily shows that if f is a cofibration in $\mathscr{C}$ and the map $\mathrm{dom}\,f'(j)\to\mathrm{cod}\,f'(j)$ is a cofibration in $\mathscr{D}$, the $f\Box f'$ is a cofibration in $\mathscr{E}$ (since $\otimes$ is a left Quillen bifunctor) which is trivial if either f or f' is trivial. The remaining claims may be found in [36]. ☐

Theorem 5.104. *Let* $\mathscr{C}$ *be a combinatorial simplicial model category and let* $\odot\colon \mathrm{sSet}\times\mathscr{C}\to\mathscr{C}$ *and* $\{-,-\}\colon \mathrm{sSet}^{\mathrm{op}}\times\mathscr{C}\to\mathscr{C}$ *denote the tensoring and cotensoring of* $\mathscr{C}$ *over* sSet*, respectively.*

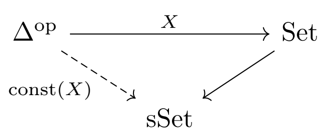
- *The homotopy colimit of* $\mathfrak{F}\colon\mathscr{D}\to\mathscr{C}$ *is given by*
$$\int^{\mathscr{D}}L_{\mathrm{inj}}(\star)\odot L_{\mathrm{proj}}(\mathfrak{F})\simeq\operatorname*{hocolim}_{\mathscr{D}}\mathfrak{F}\simeq\int^{\mathscr{D}}L_{\mathrm{proj}}(\star)\odot L_{\mathrm{inj}}(\mathfrak{F})$$
where L_{proj} *and* L_{inj} *are cofibrant resolutions for the respective model structures (note that* L_{inj} *on the far left is not the same as* L_{inj} *on the*

far right since the first corresponds to the model category $\mathrm{sSet}_{\mathrm{inj}}^{\mathscr{D}^{\mathrm{op}}}$ *and the second corresponds to* $\mathscr{C}_{\mathrm{inj}}^{\mathscr{D}}$ *- similarily for* L_{proj} *and below).*

- *The homotopy limit of* $\mathfrak{F}\colon \mathscr{D}\to\mathscr{C}$ *is given by*
$$\int_{\mathscr{D}}\{R_{\mathrm{inj}}(\star),R_{\mathrm{inj}}(\mathfrak{F})\}\simeq \operatorname*{holim}_{\mathscr{D}}\mathfrak{F}\simeq\int_{\mathscr{D}}\{R_{\mathrm{proj}}(\star),R_{\mathrm{proj}}(\mathfrak{F})\}$$
where R_{inj} *and* R_{proj} *are fibrant resolutions for the respective model structures.*

Proof. We have
$$\mathscr{C}(\star\odot X,Y)\cong \mathrm{Map}(\star,\mathscr{C}(X,Y))\cong\mathscr{C}(X,Y)$$
for all $X,Y\in\mathscr{C}$ (where $\mathrm{Map}(-,-)$ is the internal hom in sSet) and hence
$$\star\odot - \cong 1_{\mathscr{C}}$$
From this we immediately infer that $\int^{\mathscr{D}}\star\odot(-)\cong\operatorname*{colim}_{\mathscr{D}}$. Since this is a left Quillen functor, we may derive it to find:
$$\operatorname*{hocolim}_{\mathscr{D}}=\mathbb{L}\int^{\mathscr{D}}\star\odot(-)=\int^{\mathscr{D}}\star\odot L_{\mathrm{inj}}(-)$$
But then since
$$\int^{\mathscr{D}}(-)\odot(-)$$
is a left Quillen bifunctor (by Proposition 5.103), the weak equivalence $L_{\mathrm{proj}}(\star)\xrightarrow{\simeq}\star$ induces a natural weak equivalence
$$\int^{\mathscr{D}}L_{\mathrm{proj}}(\star)\odot L_{\mathrm{inj}}(-)\simeq\int^{\mathscr{D}}\star\odot L_{\mathrm{inj}}(-)$$
by Ken Brown's Lemma 5.17. Since homotopy colimits are unique up to a contractible choice, that is, up to weak equivalence, this yields the claim:
$$\operatorname*{hocolim}_{\mathscr{D}}\simeq\int^{\mathscr{D}}L_{\mathrm{proj}}(\star)\odot L_{\mathrm{inj}}(-)$$
□

Corollary 5.105. *Every simplicial set* $X\in\mathrm{sSet}$ *is the homotopy colimit over its cells. More precisely, for a simplicial set* X *we may consider the bisimplicial set* $\mathrm{const}(X)$ *which is given by the composition of functors*



where $\pi\colon\Delta\to\star$ *is the unique functor into the terminal category and* $\pi^\star$ *is the induced precomposition functor* $\mathrm{Set}^\star\cong\mathrm{Set}\to\mathrm{sSet}$*. If then* $\mathrm{sSet}_{\mathrm{inj}}^{\Delta^{\mathrm{op}}}$ *is endowed with the injective model structure (with respect to the Quillen model structure on* sSet*), then the homotopy colimit over* $\mathrm{const}(X)$ *is weakly equivalent to the original simplicial set* X*:*
$$\operatorname*{hocolim}_{\Delta^{\mathrm{op}}}\mathrm{const}(X)\simeq X$$

Proof. We have

$$\operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \operatorname{const}(X) \simeq \int^{\Delta^{\mathrm{op}}} L(\star)_n \odot X_n$$

where $L(\star)\colon \Delta \to \mathrm{sSet}$ is a cofibrant resolution of the point $\star$, that is, $L(\star)_n \xrightarrow{\simeq} \star$ and $L(\star)_n$ is cofibrant for all $n \in \mathbb{N}$. But the Yoneda embedding $よ_\Delta\colon \Delta \to \mathrm{sSet}$ is such a cofibrant resolution: The cofibrancy condition is clear, since any simplicial set is cofibrant. For the claim about the objectwise weak equivalences we note that there is a unique morphism $\Delta^n \to \star$ for every n. For the corresponding homotopy groups we then have

$$\pi_k(|\Delta^n|, x) \longrightarrow \pi_k(\star, \star) = \{0\}$$

which is an isomorphism for all $k \in \mathbb{N}$. Thus $\Delta^n \xrightarrow{\simeq} \star$ for all n. Putting all the pieces together we therefore obtain

$$\operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \operatorname{const}(X) \simeq \int^{\Delta^{\mathrm{op}}} \Delta^n \odot X_n \cong X$$

□

5.8.1. *Bar and Cobar Construction.* The following chapter is based on the corresponding section in [34].

The preceding formulas for homotopy limits and colimits do not seem very appealing. Calculating these Kan extensions concretely is nigh impossible. However, there are wondrous mathematical machineries that one may employ at this point. These are called the bar and cobar constructions.

Definition 5.106. Let $\mathscr{C}$ be a simplicially enriched tensored and cotensored category.

- The *two-sided simplicial bar construction* for small diagram functors $\mathfrak{U}\colon \mathscr{D}^{\mathrm{op}} \to \mathrm{sSet}$ and $\mathfrak{F}\colon \mathscr{D} \to \mathscr{C}$ is a simplicial object $B_\bullet(\mathfrak{U}, \mathscr{D}, \mathfrak{F})$ in $\mathscr{C}$ whose n-simplices are defined by the coproduct
$$B_n(\mathfrak{U}, \mathscr{D}, \mathfrak{F}) := \coprod_{s \in \mathscr{D}^{[n]}} \mathfrak{U}s_n \odot \mathfrak{F}s_0$$
- The *bar construction* is the geometric realization of the simplicial bar construction
$$B(\mathfrak{U}, \mathscr{D}, \mathfrak{F}) := |B_\bullet(\mathfrak{U}, \mathscr{D}, \mathfrak{F})| := \int^{\Delta^{\mathrm{op}}} \Delta^n \odot B_n(\mathfrak{U}, \mathscr{D}, \mathfrak{F})$$

Remark 5.107. The unique maps $\Delta^n \to \star$ collect into a natural transformation $よ_\Delta \to \star$. Applying the functor

$$- \odot_{\Delta^{\mathrm{op}}} B_\bullet(\mathfrak{U}, \mathscr{D}, \mathfrak{F}) := \int^{\Delta^{\mathrm{op}}} (-) \odot B_n(\mathfrak{U}, \mathscr{D}, \mathfrak{F})$$

induces a map

$$B(\mathfrak{U}, \mathscr{D}, \mathfrak{F}) \to \mathfrak{U} \odot_{\mathscr{D}} \mathfrak{F} := \int^{\mathscr{D}} \mathfrak{U} \odot \mathfrak{F}$$

where the codomain of this map is referred to as the *functor tensor product* of $\mathfrak{U}$ and $\mathfrak{F}$. In this sense, the two-sided bar construction is a fattened up version of the functor tensor product.

Example 5.108. If $\mathscr{C} = \mathrm{Set}$, then $B_\bullet(\star, \mathscr{D}, \star)$ boils down to the nerve of the category $\mathscr{D}$, that is,

$$B_n(\star, \mathscr{D}, \star) = \coprod_{s \in \mathscr{D}^{[n]}} \star \odot \star \cong \mathscr{D}^{[n]} = \mathfrak{N}\mathscr{D}_n$$

Example 5.109. Let G be a group and consider it as a one-object category. The *classifying space* of G, commonly denoted BG, is defined to be the geometric realization of the simplicial object $B_n(\star, G, \star) = G^n$.

Example 5.110. We may consider the objects $B(\mathscr{D}(-,d), \mathscr{D}, \mathfrak{F}) \in \mathscr{C}$ in the special case where $\mathfrak{U}$ is a representable functor. This depends functorially on $d \in \mathscr{D}$, so we may define a functor

$$B(\mathscr{D}, \mathscr{D}, \mathfrak{F}) \colon \mathscr{D} \to \mathscr{C}, \qquad d \mapsto B(\mathscr{D}(-,d), \mathscr{D}, \mathscr{F})$$

In particular, by allowing $\mathfrak{F}$ to vary we obtain a functor $B(\mathscr{D}, \mathscr{D}, -) \colon \mathscr{C}^{\mathscr{D}} \to \mathscr{C}^{\mathscr{D}}$.

Lemma 5.111. *We have natural isomorphisms*

$$B_\bullet(\mathscr{D}(-,d), \mathscr{D}, \star) \cong \mathfrak{N}(\mathscr{D}/d), \qquad B_\bullet(\star, \mathscr{D}, \mathscr{D}(d,-)) \cong \mathfrak{N}(d/\mathscr{D})$$

and hence, in particular, there are natural isomorphisms $B_\bullet(\mathscr{D}, \mathscr{D}, \star) \colon \mathfrak{N}(\mathscr{D}/-) \colon \mathscr{D} \to \mathrm{sSet}$ *and* $B_\bullet(\star, \mathscr{D}, \mathscr{D}) \cong \mathfrak{N}(-/\mathscr{D}) \colon \mathscr{D}^{\mathrm{op}} \to \mathrm{sSet}$.

Proof. We have to establish a natural bijection

$$\mathfrak{N}(\mathscr{D}/d)_n = \mathrm{Fun}([n], \mathscr{D}/d) \to B_n(\mathscr{D}(-,d), \mathscr{D}, \star) = \coprod_{s \in \mathscr{D}^{[n]}} \mathscr{D}(s_n, d)$$

A functor $f \colon [n] \to \mathscr{D}/d$ is nothing more than the information of a commutative diagram

$$\begin{array}{ccccccccc} s_0 & \xrightarrow{f_1} & s_1 & \xrightarrow{f_2} & s_2 & \longrightarrow & \dots & \xrightarrow{f_n} & s_n \\ & & & & & & & & \big\downarrow \tilde{f} \\ & & & & & & & & d \end{array}$$

But this is completely determined by only the information of the n-tuple of arrows $s_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} s_n$ along with the map $s_n \xrightarrow{\tilde{f}} d$. Thus, by forgetting the dotted arrows above, we obtain a natural bijection

$$\mathfrak{N}(\mathscr{D}/d)_n \to B_n(\mathscr{D}(-,d), \mathscr{D}, \star)$$

The other isomorphism is constructed analogously. $\square$

We shall dualize the bar construction to arrive at the cobar construction. Before doing that, let us introduce some new notions. In the presence of a cotensor $\{-,-\} \colon \mathrm{sSet}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$, the *functor cotensor product* or sometimes *functor hom* of $\mathfrak{U} \colon \mathscr{D} \to \mathrm{sSet}$ and $\mathfrak{F} \colon \mathscr{D} \to \mathscr{C}$ is the end

$$\{\mathfrak{U}, \mathfrak{F}\}^{\mathscr{D}} := \int_{d \in \mathscr{D}} \{\mathfrak{U}d, \mathfrak{F}d\}$$

Then if $\mathscr{C}$ is cotensored over sSet, by means of $\{-,-\}$, then the totalization of a cosimplicial object $X^\bullet \colon \Delta \to \mathscr{C}$ is defined by

$$\mathrm{Tot} X^\bullet := \{ よ_\Delta, X^\bullet\}^\Delta = \int_\Delta \{\Delta^n, X^n\}$$

The cosimplicial cobar construction is then a fattened up version of the functor cotensor product.

Definition 5.112. Let $\mathscr{C}$ be complete and cotensored over sSet and let $\mathfrak{F}\colon \mathscr{D} \to \mathscr{C}$ and $\mathfrak{U}\colon \mathscr{D} \to \mathrm{sSet}$ be small diagram functors.

- The *cosimplicial cobar construction* $C^\bullet(\mathfrak{U}, \mathscr{D}, \mathfrak{F})$ is a cosimplicial object in $\mathscr{C}$ which has n-simplices
$$C^n(\mathfrak{U}, \mathscr{D}, \mathfrak{F}) := \prod_{s \in \mathscr{D}^{[n]}} \{\mathfrak{U}s_0, \mathfrak{F}s_n\}$$
- The *cobar construction* $C(\mathfrak{U}, \mathscr{D}, \mathfrak{F})$ is the totalization of the cosimplicial cobar construction, i.e.,
$$C(\mathfrak{U}, \mathscr{D}, \mathfrak{F}) := \{ よ_\Delta, C^\bullet(\mathfrak{U}, \mathscr{D}, \mathfrak{F})\}^\Delta = \int_\Delta \{\Delta^n, C^n(\mathfrak{U}, \mathscr{D}, \mathfrak{F})\}$$

Theorem 5.113. *Let $\mathscr{C}$ be a simplicially enriched model category with cofibrant replacement functor L and fibrant replacement functor R.*

- *The functor*
$$B(\mathscr{D}, \mathscr{D}, L-)\colon \mathscr{C}^{\mathscr{D}} \to \mathscr{C}^{\mathscr{D}}$$
gives rise to a left deformation for $\operatorname*{colim}_{\mathscr{D}}\colon \mathscr{C}^{\mathscr{D}} \to \mathscr{C}$.
- *The functor*
$$B(\mathscr{D}, \mathscr{D}, R-)\colon \mathscr{C}^{\mathscr{D}} \to \mathscr{C}^{\mathscr{D}}$$
gives rise to a right deformation for $\lim_{\mathscr{D}}\colon \mathscr{C}^{\mathscr{D}} \to \mathscr{C}$.

Proof. See Theorem 5.1.1. in [34]. □

Corollary 5.114. *If $\mathscr{C}$ is a simplicial model category and $\mathscr{D}$ is any small category, then the functors* $\operatorname*{colim}_{\mathscr{D}}, \lim_{\mathscr{D}}\colon \mathscr{C}^{\mathscr{D}} \to \mathscr{C}$ *admit left and right derived functors, denoted* $\operatorname*{hocolim}_{\mathscr{D}}$ *and* $\operatorname*{holim}_{\mathscr{D}}$*, which are given by*
$$\operatorname*{hocolim}_{\mathscr{D}} := \mathbb{L}\operatorname*{colim}_{\mathscr{D}} \simeq B(\star, \mathscr{D}, L-), \qquad \operatorname*{holim}_{\mathscr{D}} := \mathbb{R}\lim_{\mathscr{D}} \simeq C(\star, \mathscr{D}, R-)$$

Proof. By Theorem 5.113 and Theorem 5.66 we have
$$\mathbb{L}\operatorname*{colim}_{\mathscr{D}} \simeq \operatorname*{colim}_{\mathscr{D}} B(\mathscr{D}, \mathscr{D}, L-) \simeq \star \odot_{\mathscr{D}} B(\mathscr{D}, \mathscr{D}, L-)$$
But then
$$\star \odot_{\mathscr{D}} B(\mathscr{D}, \mathscr{D}, L-) \simeq B(\star \odot_{\mathscr{D}} \mathscr{D}, \mathscr{D}, L-) \simeq B(\star, \mathscr{D}, L-)$$
(for details see [34] Corollary 5.1.3.). Analogously,
$$\mathbb{R}\lim_{\mathscr{D}} \simeq \lim_{\mathscr{D}} C(\mathscr{D}, \mathscr{D}, R-) \simeq \{\star, C(\mathscr{D}, \mathscr{D}, R-)\}^{\mathscr{D}} \simeq C(\star, \mathscr{D}, R-)$$
□

Example 5.115. The homotopy colimit of the terminal functor $\star\colon \mathscr{D} \to \mathscr{C}$ is $B(\star, \mathscr{D}, \star)$, which is isomorphic to the geometric realization of the nerve of $\mathscr{D}$. In the case where $\mathscr{D}$ is a 1-object groupoid, that is, a group G, this space
$$BG := B(\star, G, \star) \cong B(\star \odot_G G, G, \star) \cong \star \odot_G B(G, G, \star) =: \operatorname*{colim}_G EG$$
is called the classifying space of G. More generally, $B(\star, \mathscr{D}, \star)$ is referred to as the *classifying space* of the category $\mathscr{D}$.

Theorem 5.116 ([34] Theorem 6.6.1)**.** *Let $\mathfrak{F}\colon \mathscr{D} \to \mathscr{C}$ be any diagram in a complete and cocomplete, tensored, cotensored and simplicially enriched category $\mathscr{C}$. Then there are natural isomorphisms*

$$B(\star, \mathscr{D}, \mathfrak{F}) \cong \mathfrak{N}(-/\mathscr{D}) \odot_{\mathscr{D}} \mathfrak{F}, \qquad C(\star, \mathscr{D}, \mathfrak{F}) \cong \{\mathfrak{N}(\mathscr{D}/-), \mathfrak{F}\}^{\mathscr{D}}$$

In particular, the homotopy colimit of a pointwise cofibrant diagram $\mathfrak{F}$ can be computed by the functor tensor product with $\mathfrak{N}(-/\mathscr{D})$. Dually the homotopy limit of a pointwise fibrant diagram can be computed by the functor cotensor product with $\mathfrak{N}(\mathscr{D}/-)$.

Proof. We will prove the result by means of some coend calculus. By Fubini's Theorem for coends and cocontinuity of simplicial tensors we get:

$$\begin{aligned}
\mathfrak{N}(-/\mathscr{D}) \odot_{\mathscr{D}} \mathfrak{F} &\cong B(\star, \mathscr{D}, \mathscr{D}) \odot_{\mathscr{D}} \mathfrak{F} \\
&\cong \int^{d\in\mathscr{D}} |B_\bullet(\star, \mathscr{D}, \mathscr{D}(d,-))| \odot \mathfrak{F}d \\
&\cong \int^{d\in\mathscr{D}} \Big(\int^{[n]\in\Delta} \Delta^n \times B_n(\star, \mathscr{D}, \mathscr{D}(d,-)) \Big) \odot \mathfrak{F}d \\
&\cong \int^{d\in\mathscr{D}} \int^{[n]\in\Delta} \Delta^n \odot (B_n(\star, \mathscr{D}, \mathscr{D}(d,-)) \odot \mathfrak{F}d) \\
&\cong \int^{d\in\mathscr{D}} \Delta^n \odot \Big(\int^{[n]\in\Delta} B_n(\star, \mathscr{D}, \mathscr{D}(d,-)) \odot \mathfrak{F}d \Big)
\end{aligned}$$

Moreover,

$$\begin{aligned}
B(\star, \mathscr{D}, \mathfrak{F}) &\cong \int^{[n]\in\Delta} \Delta^n \odot B_n(\star, \mathscr{D}, \mathfrak{F}) \\
&\cong \int^{[n]\in\Delta} \Delta^n \odot \Big(\coprod_{s\in\mathscr{D}^{[n]}} \mathfrak{F}s_0 \Big)
\end{aligned}$$

Hence we simply have to prove that we have an isomorphism

$$\int^{d\in\mathscr{D}} B_n(\star, \mathscr{D}, \mathscr{D}(d,-)) \odot \mathfrak{F}d \cong \coprod_{s\in\mathscr{D}^{[n]}} \mathfrak{F}s_0$$

By Lemma 5.111 we have $B_n(\star, \mathscr{D}, \mathscr{D}(d,-)) \cong \mathfrak{N}(d/\mathscr{D})_n$, so the LHS of may be rewritten as

$$\int^{d\in\mathscr{D}} \coprod_{\mathfrak{N}(d/\mathscr{D})_n} \mathfrak{F}d$$

But elements in $\mathfrak{N}(d/\mathscr{D})_n$ are strings $s\colon [n] \to \mathscr{D}$ of n composable morphisms in $\mathscr{D}$ together with an arrow $d \to s_0$ in $\mathscr{D}$. Thus, we obtain

$$\int^{d\in\mathscr{D}} \coprod_{s\in\mathscr{D}^{[n]}} \mathscr{D}(d, s_0) \odot \mathfrak{F}d \cong \coprod_{d\in\mathscr{D}^{[n]}} \int^{d\in\mathscr{D}} \mathscr{D}(d, s_0) \odot \mathfrak{F}d \cong \coprod_{s\in\mathscr{D}^{[n]}} \mathfrak{F}s_0$$

as wanted. □

For more details on homotopy (co)limits and categorical homotopy theory, see [34].

6. Sheaf Theory and Localizations

> I have no use for adventures. Nasty disturbing uncomfortable things! Make you late for dinner!
>
> Tolkien, J.R.R. The Hobbit (Bilbo Baggins)

This section is based on [9] and the corresponding Nlab article on Sheaves.

The study of sheaves is, at its core, the study of *locality*, which captures the idea that the properties of an object can be understood by examining it locally. Sheaves provide a powerful framework for encoding this local information and gluing it together to obtain a global understanding. However, in certain situations, the conventional framework of sheaves falls short, and a more flexible and sophisticated notion is required to capture the intricate nature of the data under consideration. This is where the notion of ∞-sheaves comes into play. An ordinary sheaf is a mathematical construct that assigns local data to each open subset of a topological space. The concept of locality is encoded through the sheaf condition, which states that the local data on overlapping open subsets should be compatible. However, in modern mathematics, there is a growing need to handle more complex and nuanced data that goes beyond what can be captured by ordinary sheaves. To address this limitation, the theory of ∞-sheaves was developed. The key idea behind *∞-sheaves* is to generalize the notion of locality to account for higher-dimensional information. Instead of assigning ordinary sets or groups to open subsets, ∞-sheaves assign higher categorical structures, such as ∞-groupoids or ∞-categories. These higher structures allow for a more refined encoding of local data, capturing not only the objects themselves but also the rich network of relationships and interactions between them. In this chapter we will first discuss the notion of ordinary sheaves on a *site* and we will investigate how any such category of sheaves is really just a reflective subcategory embedding. This in turn will motivate the notion of ∞-sheaf, as this will be a homotopical version of a reflective subcategory embedding.

6.1. **(Pre)Sheaves.** Let us start off this chapter by reminding the reader of the classical definition of a presheaf on a topological space. Fix a topological space X. Then X induces a poset category of open subsets $\mathcal{O}X$ which has as its objects the open subsets of X and morphisms are inclusions. A presheaf $\mathfrak{S}$ on $\mathcal{O}X$ (or put differently a functor $\mathfrak{S}\colon \mathcal{O}X^{\mathrm{op}} \to \mathrm{Set}$) then boils down to providing a *family of sections* $(\mathfrak{S}U)_{U\in\mathcal{O}X}$ and *restriction maps* $|_V\colon \mathfrak{S}U \to \mathfrak{S}V$ for every inclusion $V \subset U$ in $\mathcal{O}X$. The canonical example of such a presheaf is the hom-functor $よ_{\mathrm{Top}}X = \mathrm{Top}(-,X)$ which takes an open subset $U \subset X$ to the set of continuous functions $\mathrm{Top}(U,X)$ and an inclusion $V \subset U$ is taken to the genuine restriction map $|_V\colon \mathrm{Top}(U,X) \to \mathrm{Top}(V,X)$. We then also readily notice that $\mathrm{Top}(-,X)$ yields a *sheaf*, that is, it satisfies *Serre's condition*: Given any $U \in \mathcal{O}X$ and any open cover $(U_i)_{i\in I}$ of U such that whenever there are maps $f_i \in \mathrm{Top}(U_i,X)$ subject to the condition

$$f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$$

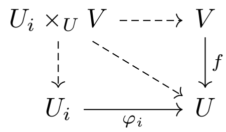
for all $i,j \in I$, then there exists exactly one element $f \in \mathrm{Top}(U,X)$ such that $f|_{U_i} = f$ for all $i \in I$. In other words, continuity is, unsurprisingly, really a purely local property. Generalizing the notion of presheaf is quite straightforward then.

Definition 6.1. A *presheaf* on a category $\mathscr{C}$ is a functor $\mathfrak{S}\colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$.

However, generalizing the notion of a sheaf to something more general than the category $\mathcal{O}X$ takes a little more work:

Definition 6.2. Let $\mathscr{C}$ be a category.

- A *Grothendieck topology* on $\mathscr{C}$ is a set $\mathscr{J}$ of families of morphisms $\{\varphi_i : U_i \to U\}$ with common codomains (known as *coverings*) such that
  - for any isomorphism φ in $\mathscr{C}$ we have $\{\varphi\} \in \mathscr{J}$.
  - if $\{U_i \to U\} \in \mathscr{J}$ and $\{V_{ij} \to U_i\} \in \mathscr{J}$ for each i, then the family of respective compositions $\{V_{ij} \to U\}$ is in $\mathscr{J}$.
  - if $\{U_i \xrightarrow{\varphi_i} U\} \in \mathscr{J}$ and $f: V \to U$ is any morphism, then the pullback



  exists and $\{U_i \times_U V \to V\} \in \mathscr{J}$.
- A *Grothendieck site* or just *site* $(\mathscr{C}, \mathscr{J})$ is a small category $\mathscr{C}$ endowed with a Grothendieck topology $\mathscr{J}$.

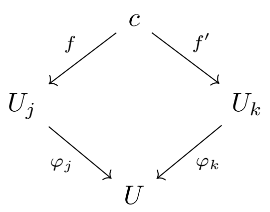
The notion of a Grothendieck topology generalizes the notion of open covers of topological spaces. The respective axioms for the set $\mathscr{J}$ in the above definition are also quite natural: The only isomorphism in $\mathcal{O}X$ is the identity on X itself (where X is assumed to be a topological space). Certainly, $\{X \hookrightarrow X\}$ itself yields an open cover for X. Thus, for any general isomorphism $\varphi\colon \mathrm{dom}\varphi \to \mathrm{cod}\varphi$, it is natural to assume that $\{\varphi\}$ constitutes a cover for $\mathrm{cod}\varphi$. Moreover, the second axiom in the definition of a Grothendieck topology $\mathscr{J}$ is certainly also something which holds for open covers of topological spaces and therefore it is natural to also assume this condition to hold true for a general cover. Lastly, if we consider the pullback diagram in the above definition with respect to an open cover for a topological space X, then $U_i \times_X V$ in $\mathcal{O}X$ is nothing else than $U_i \cap V$ and $U_i \cap V \to U_i$ is simply an inclusion, that is, an element of the open cover.

Definition 6.3. Let $(\mathscr{C}, \mathscr{J})$ be a (small) site. A presheaf $\mathfrak{S} \in \mathrm{Psh}(\mathscr{C})$ is called a *sheaf* (or $\mathscr{J}$*-sheaf*) if

- for every covering family $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ in $\mathscr{J}$,
- and for every *compatible family* of elements given by a collection

$$(s_i \in \mathfrak{S}U_i)_{i \in I}$$

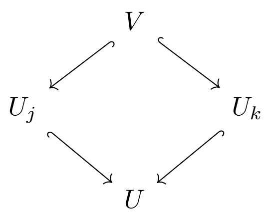
  such that for all $j, k \in I$ and all morphisms $U_j \xleftarrow{f} c \xrightarrow{f'} U_k$ in $\mathscr{C}$ so that



  commutes, we have

$$\mathfrak{S}(f)(s_j) = \mathfrak{S}(f')(s_k) \in \mathfrak{S}c$$

Remark 6.4. The category $\mathcal{O}X$ is certainly a site, where the corresponding Grothendieck topology is given by the collections of open covers of objects in $\mathcal{O}X$. In other words, a covering family $\{U_i \to U\}$ in $\mathcal{O}X$ is simply a family of open subsets of U such that $\cup U_i = U$. A presheaf $\mathfrak{S}\colon \mathcal{O}X^{\mathrm{op}} \to \mathrm{Set}$ is then a sheaf precisely if for every covering family $\{U_i \hookrightarrow U\}_{i\in I}$ and any compatible family of elements $(s_i \in \mathfrak{S}U_i)_{i\in I}$ such that for all $j,k \in I$ and all $U_j \hookleftarrow V \hookrightarrow U_k$ in $\mathscr{O}X$ so that



commutes (which boils down to $V \subset U_j \subset U$ and $V \subset U_k \subset U$ respectively), we have

$$s_i|_V = s_j|_V$$

which precisely agrees with the usual definition of sheaf on $\mathscr{O}X$.

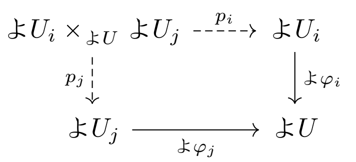
We shall next give an equivalent definition of what a sheaf is in more general abstract terms:

Definition 6.5. Let $(\mathscr{C}, \mathscr{J})$ be a site.

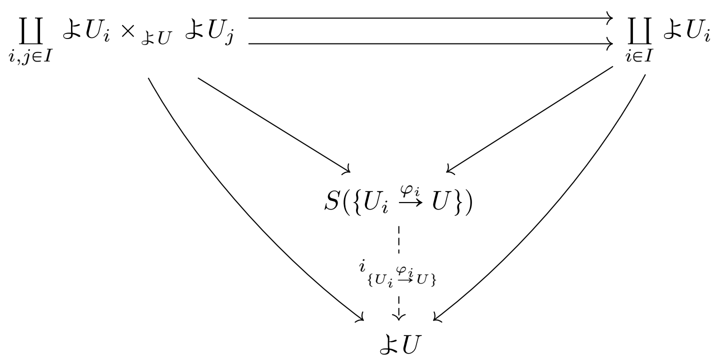
- Given a covering $\{U_i \xrightarrow{\varphi_i} U\}_{i\in I}$ in $\mathscr{J}$, its corresponding *Čech sieve* is the presheaf $S(\{U_i \xrightarrow{\varphi_i} U\})$ defined as the coequalizer of the diagram

$$\coprod_{i,j\in I} \text{よ}U_i \times_{\text{よ}U} \text{よ}U_j \rightrightarrows \coprod_{i\in I} \text{よ}U_i$$

  where $\text{よ}\colon \mathscr{C} \to \mathrm{Psh}(\mathscr{C})$ is the Yoneda embedding. Here the coproduct on the left of the above diagram is defined via the pullbacks



  while the two parallel arrows are those induced componentwise by the two projections p_i and p_j in the pullback diagram.
- Denote by $i_{\{U_i \xrightarrow{\varphi_i} U\}}\colon S(\{U_i \xrightarrow{\varphi_i} U_i\}) \to \text{よ}U$ the canonical morphism that is induced by the universal property of the coequalizer from the morphisms $\text{よ}\varphi_i\colon \text{よ}U_i \to \text{よ}U$ and $\text{よ}U_i \times_{\text{よ}U} \text{よ}U_j \to \text{よ}U$:



Remark 6.6. Since (co)limits are calculated componentwise in a category of presheaves, we may calculate the sieve $S(\{U_i \xrightarrow{\varphi_i} U\})$ very explicitly. Indeed, recall that the colimit of a diagram $D\colon \mathscr{D} \to \mathrm{Set}$ may be computed as

$$\operatorname*{colim}_{d\in\mathscr{D}} Dd \cong \frac{\coprod Dd}{\sim}$$

where $\sim$ is an equivalence relation on the set $\coprod_{d\in\mathscr{D}} Dd$ defined by

$$(d, t \in Dd) \sim (d', t' \in Dd') \overset{\text{def}}{\iff} \exists f\colon d \to d'\colon D(f)(t) = t'$$

Thus, if $c \in \mathscr{C}$ is a fixed object then we may consider the diagram $S(\{U_i \xrightarrow{\varphi_i} U\})(c)$ given by the coequalizer of

$$\coprod_{i,j\in I} \mathscr{C}(c, U_i) \times_{\mathscr{C}(c,U)} \mathscr{C}(c, U_j) \rightrightarrows \coprod_{i\in I} \mathscr{C}(c, U_i)$$

Using the above formula for general colimits of set-valued diagrams we deduce that $S(\{U_i \xrightarrow{\varphi_i} U\})(c)$ corresponds to those morphisms $f \in \mathscr{C}(c, U)$ that factor through some φ_k for some $k \in I$:

$$\begin{array}{ccc} c & \overset{\exists f_k}{\dashrightarrow} & U_k \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \varphi_k} & \\ & U & \end{array}$$

Definition 6.7. A *sheaf* on a site $(\mathscr{C}, \mathscr{J})$ or a $\mathscr{J}$*-sheaf* is a presheaf $\mathfrak{S} \in \mathrm{Psh}(\mathscr{C})$ that is a *local object* with respect to all $i_{\{U_i \xrightarrow{\varphi_i} U\}}$: an object such that for all covering families $\{U_i \xrightarrow{\varphi_i} U\}$ in $\mathscr{J}$ we have that the hom-functor sends the canonical morphisms $i_{\{U_i \xrightarrow{\varphi_i} U\}}\colon S(\{U_i \xrightarrow{\varphi_i} U\}) \to \text{よ}U$ to isomorphisms:

$$\mathrm{Psh}_{\mathscr{C}}(\text{よ}U, \mathfrak{S}) \xrightarrow{i^\star_{\{U_i \xrightarrow{\varphi_i} U\}}} \mathrm{Psh}_{\mathscr{C}}(S(\{U_i \xrightarrow{\varphi_i} U\}), \mathfrak{S})$$

The *category of* $\mathscr{J}$*-sheaves* $\mathrm{Sh}_{\mathscr{J}}(\mathscr{C})$ or $\mathrm{Sh}_{(\mathscr{C},\mathscr{J})}$ is the full subcategory of presheaves which has only $\mathscr{J}$-sheaves as its objects.

Remark 6.8. The above can be reformulated by means of the Yoneda Lemma and the fact that the contravariant functor $\mathrm{Psh}_{\mathscr{C}}(-, \mathfrak{S})$ sends colimits to limits: A presheaf $\mathfrak{S}$ is then a sheaf if and only if the induced diagram

$$\mathfrak{S}U \dashrightarrow \coprod_{i\in I} \mathrm{Psh}_{\mathscr{C}}(\text{よ}U_i, \mathfrak{S}) \rightrightarrows \coprod_{i,j\in I} \mathrm{Psh}_{\mathscr{C}}(\text{よ}U_i \times_{\text{よ}U} \text{よ}U_j,\ \mathfrak{S})$$

is an equalizer diagram for each covering family $\{U_i \to X\} \in \mathscr{V}$. Thus, since the pullbacks of presheaves $\text{よ}U_i \times_{\text{よ}U} \text{よ}U_j$ are themselves representable by definition of a site, we know that the pullback $U_i \times_U U_j$ exists in $\mathscr{C}$ even before passing to the Yoneda embedding. Hence applying the Yoneda Lemma the sheaf condition boils down to

$$\mathfrak{S}U \dashrightarrow \prod_{i\in I} \mathfrak{S}(U_i) \rightrightarrows \prod_{i,j\in I} \mathfrak{S}(U_i \times_U U_j)$$

being an equalizer diagram. This is referred to as the *descent condition* along the covering family.

Proposition 6.9. *The condition that the induced morphism*

$$i^\star_{\{U_i \xrightarrow{\varphi_i} U\}}$$

is an isomorphism for a cover $\{U_i \xrightarrow{\varphi_i} U\}$ *is equivalent to the condition that the set* $\mathfrak{S}U$ *is isomorphic (in bijective correspondence) to the set of compatible families* $(s_i \in \mathfrak{S}U_i)$ *as given in Definition 6.3.*

Proof. See the Nlab Sheaf Proposition 2.8. □

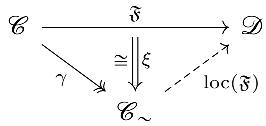
6.2. **Reflective Localization.** Recall the notions of *category with weak equivalences* and *localization of a category* as defined in chapter 5.2. We have explicitly constructed the localization at a class of weak equivalences $\mathscr{W} \subset \mathscr{C}$. However, one may also go about the definition in a more abstract manner:

Definition 6.10. Let $\mathscr{C}$ be a category with weak equivalences $\mathscr{W}$. Then the localization of $\mathscr{C}$ at $\mathscr{W}$ is, if it exists,

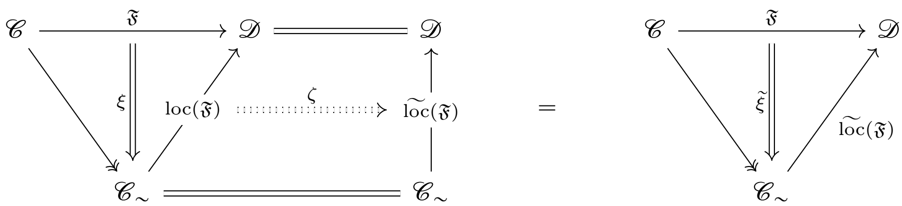
- a category $\mathscr{C}[\mathscr{W}^{-1}]$,
- a functor $\gamma\colon \mathscr{C} \twoheadrightarrow \mathscr{C}[\mathscr{W}^{-1}] =: \mathscr{C}_\sim$

such that

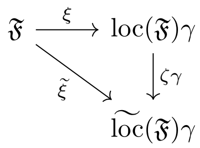
- γ sends all morphisms in $\mathscr{W}$ to isomorphisms in $\mathscr{C}_\sim$.
- γ is universal with this property: If $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ is any functor which sends morphisms in $\mathscr{W}$ to isomorphisms, then $\mathfrak{F}$ factors through γ up to natural isomorphism:



and any two such factorizations $\mathrm{loc}(\mathfrak{F})$ and $\widetilde{\mathrm{loc}}(\mathfrak{F})$ are related by a unique natural isomorphism $\zeta\colon \mathrm{loc}(\mathfrak{F}) \to \widetilde{\mathrm{loc}}(\mathfrak{F})$ compatible with $\xi\colon \mathfrak{F} \xrightarrow{\cong} \mathrm{loc}(\mathfrak{F})\gamma$ and $\tilde{\xi}\colon \mathfrak{F} \xrightarrow{\cong} \widetilde{\mathrm{loc}}(\mathfrak{F})\gamma$:

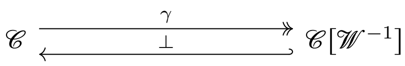


In other words, we have a commutative diagram



Remark 6.11. The previous definition certainly extends the notion of localization of a category $\mathscr{C}$ at a class of weak equivalences $\mathscr{W}$. Our original definition of course satisfies all the respective properties (the involved natural isomorphisms are just identities).

Definition 6.12. Let $\mathscr{C}$ be a category with weak equivalences $\mathscr{W}$. Then the localization of $\mathscr{C}$ at $\mathscr{W}$ is called a *reflective localization*, if the localization functor has a fully faithful right adjoint, exhibiting it as the reflection functor of a reflective subcategory-inclusion:



Definition 6.13. Let $\mathscr{C}$ be a category and let $S \subset \mathrm{Mor}\mathscr{C}$ be a set of morphisms in $\mathscr{C}$.

- An object $c \in \mathscr{C}$ is called *S-local* if for all $s \in S$ the hom-functor induces a bijection

$$\mathscr{C}(\mathrm{cod}(s), c) \xrightarrow{\mathscr{C}(s,c)} \mathscr{C}(\mathrm{dom}(s), c)$$

  In other words, any morphism $f\colon \mathrm{dom}(s) \to c$ extends uniquely along s to $\mathrm{cod}(s)$:

$$\begin{array}{ccc} \mathrm{dom}(s) & \xrightarrow{f} & c \\ {\scriptstyle s}\downarrow & \nearrow_{\exists !} & \\ \mathrm{cod}(s) & & \end{array}$$

- A morphism f in $\mathscr{C}$ is an *S-local morphism* if for every S-local object $c \in \mathscr{C}$ the induced hom-functor

$$\mathscr{C}(\mathrm{cod}f, c) \xrightarrow{\mathscr{C}(f,c)} \mathscr{C}(\mathrm{dom}f, c)$$

  is a bijection.
- Denote by $\iota\colon \mathscr{C}_S \hookrightarrow \mathscr{C}$ the inclusion of the full subcategory of S-local objects.
- The *reflection onto S-local objects* is, if it exists, a left adjoint L to the full subcategory inclusion $\iota\colon \mathscr{C}_S \hookrightarrow \mathscr{C}$:

$$\mathscr{C} \underset{\iota}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathscr{C}_S$$

Lemma 6.14. *Let us consider adjoint functors*

$$\mathscr{C} \underset{\mathfrak{U}}{\overset{\mathfrak{F}}{\underset{\perp}{\rightleftarrows}}} \mathscr{D}$$

Then $\mathfrak{F}$ is fully faithful if and only if the adjunction unit $\eta\colon 1_{\mathscr{C}} \to \mathfrak{U}\mathfrak{F}$ is a natural isomorphism. Moreover, if $\mathfrak{F}$ is fully faithful, then $\mathfrak{U}\varepsilon$ is a natural isomorphism.

Proof. Let us denote by φ the adjunction isomorphism

$$\mathscr{D}(\mathfrak{F}, -) \xrightarrow[\varphi]{\cong} \mathscr{C}(-, \mathfrak{U})$$

Then we have $\eta = \varphi(1_{\mathfrak{F}})$. We shall then verify that the dashed arrow is equal to the composition of the other two arrows in the diagram

$$\begin{array}{ccc} \mathscr{C}(c, c') & \xrightarrow{\mathfrak{F}} & \mathscr{D}(\mathfrak{F}c, \mathfrak{F}c') \\ & \searrow^{(\eta_{c'})_\star} & \cong \downarrow \varphi \\ & & \mathscr{C}(c, \mathfrak{U}\mathfrak{F}c') \end{array}$$

If we manage to show this, then the first of the above claims obviously holds. By naturality of φ we have a commutative diagram

$$\begin{array}{ccc} \mathscr{D}(\mathfrak{F}c', \mathfrak{F}c') & \xrightarrow[\cong]{\varphi} & \mathscr{C}(c', \mathfrak{U}\mathfrak{F}c') \\ {\scriptstyle (\mathfrak{F}f)^\star}\downarrow & & \downarrow{\scriptstyle f^\star} \\ \mathscr{D}(\mathfrak{F}c, \mathfrak{F}c') & \xrightarrow[\varphi]{\cong} & \mathscr{C}(c, \mathfrak{U}\mathfrak{F}c') \end{array}$$

Evaluating the above square at the morphism $1_{\mathfrak{F}c'}$ yields the first claim. On the other hand, if $\mathfrak{F}$ is fully faithful then we have shown that $\eta\colon 1_{\mathscr{C}} \to \mathfrak{U}\mathfrak{F}$ is a natural isomorphism. By the triangle identities we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{U} & \xrightarrow{\eta\mathfrak{U}} & \mathfrak{U}\mathfrak{F}\mathfrak{U} \\ & \searrow^{=} & \downarrow{\scriptstyle \mathfrak{U}\varepsilon} \\ & & \mathfrak{U} \end{array}$$

and hence $\mathfrak{U}\varepsilon$ is a left inverse for the natural isomorphism $\eta\mathfrak{U}$, which in turn yields that $\mathfrak{U}\varepsilon$ must be a natural isomorphism. □

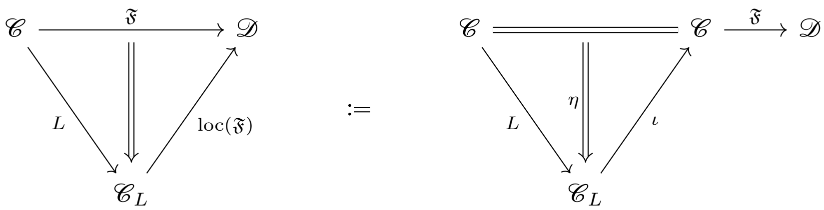
Remark 6.15. Certainly enough the above Lemma has its dual counterpart.

Proposition 6.16. *Every reflective subcategory inclusion*

$$\mathscr{C}_L \underset{\iota}{\overset{L}{\underset{\bot}{\leftrightarrows}}} \mathscr{C}$$

is the reflective localization at the class $\mathscr{W} := L^{-1}(\mathrm{Isos})$ *of morphisms that are sent to isomorphisms by the reflector* L.

Proof. We just have to verify the universal property as given in Definition 6.10. So, let $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ be a functor that maps morphisms in $\mathscr{W}$ to isomorphisms in $\mathscr{D}$. We shall first show that $\mathfrak{F}$ factors through L up to natural isomorphism. Let $\eta\colon 1_{\mathscr{C}_L} \to \iota L$ and $\varepsilon\colon L\iota \to 1_{\mathscr{C}}$ be the corresponding adjunction unit and counit, and consider the whiskering $\xi := \mathfrak{F}\eta$ along with $\mathrm{loc}(\mathfrak{F}) := \mathfrak{F}\iota$:



But $\mathfrak{F}\eta\colon \mathfrak{F} \to \mathfrak{F}\iota L$ is a natural isomorphism (by the dual counterpart of Lemma 6.14), so the factorization follows. For uniqueness up to isomorphism of this factorization see the Nlab page reflective localization Proposition 3.1. □

Proposition 6.17. *Let* $\mathscr{C}$ *be a category with weak equivalences* $\mathscr{W}$. *If the localization of* $\mathscr{C}$ *at* $\mathscr{W}$ *is reflective*

$$\mathscr{C} \underset{\iota}{\overset{L}{\underset{\bot}{\rightleftarrows}}} \mathscr{C}[\mathscr{W}^{-1}]$$

(where we use the letter L *instead of* γ *to denote the localization functor) then* $\mathscr{C}[\mathscr{W}^{-1}] \overset{\iota}{\hookrightarrow} \mathscr{C}$ *is equivalently the inclusion of the full subcategory of* $\mathscr{W}$*-local objects and hence* L *is equivalently the reflection onto* $\mathscr{W}$*-local objects.*

Proof. We shall prove that

- every object $\iota c \in \mathscr{C}$ for $c \in \mathscr{C}[\mathscr{W}^{-1}]$ is an S-local object.
- $c \in \mathscr{C}$ is $\mathscr{W}$-local if and only if it is in the essential image of $\mathscr{C}[W^{-1}] \overset{\iota}{\hookrightarrow} \mathscr{C}$.

The first claim is immediate from the natural isomorphism of functors

$$\mathscr{C}(-, \iota c) \cong \mathscr{C}[\mathscr{W}^{-1}](L, c)$$

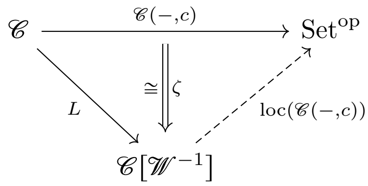
Regarding the second claim: Assume first that $c \in \mathscr{C}$ is $\mathscr{W}$-local. Our first observation is then that c must also be local with respect to the *saturated class of morphisms*

$$\mathscr{W}_{\mathrm{sat}} := L^{-1}(\mathrm{Isos})$$

that are inverted by L. Indeed, the hom-functor

$$\mathscr{C} \overset{\mathscr{C}(-,c)}{\to} \mathrm{Set}^{\mathrm{op}}$$

takes morphisms in $\mathscr{W}$ to isomorphisms (by assumption). Hence by the universal property of the localization functor L it must factor through L up to natural isomorphism:



But this implies that any morphism that is inverted by L must also be inverted by $\mathscr{C}(-,c)$, as claimed. We then know that the component η_c at c of the adjunction unit $\eta\colon 1_{\mathscr{C}} \to \iota L$ must be in $\mathscr{W}_{\mathrm{sat}}$ ($L\eta$ is a natural isomorphism by Lemma 6.14). However, $\eta_c \in \mathscr{W}_{\mathrm{sat}}$ ensures that we have a bijection

$$\mathscr{C}(\iota Lc, c) \xrightarrow[\cong]{\mathscr{C}(\eta_c, c)} \mathscr{C}(c,c)$$

Denote by η_c^{-1} the preimage of the identity morphism 1_c under the above bijection. By construction η_c^{-1} is a left inverse for η_c and by the 2-out-of 3 property $\eta_c^{-1} \in \mathscr{W}_{\mathrm{sat}}$. Moreover, the above also showed that ιLc is in $\mathscr{W}_{\mathrm{sat}}$, so we may play the same game with η_c^{-1}: We consider the bijection

$$\mathscr{C}(c, \iota Lc) \xrightarrow[\cong]{\mathscr{C}(\eta_c^{-1}, \iota Lc)} \mathscr{C}(\iota Lc, \iota Lc)$$

to obtain a left inverse $(\eta_c^{-1})^{-1}$ for η_c^{-1}. But this implies that $\eta_c\colon c \to \iota Lc$ is an isomorphism, which proves that c is in the essential image of ι. Conversely, if $c \in \mathscr{C}$ is in the essential image of ι, then it is immediate that c is $\mathscr{W}$-local. □

Proposition 6.18. *Let $\mathscr{C}$ be a category and let $S \subset \mathrm{Mor}\mathscr{C}$ be a class of morphisms in $\mathscr{C}$. Then the reflection onto S-local objects (Definition 6.13) satisfies, if it exists, the universal property of a localization of categories with respect to left adjoint functors inverting morphisms in S.*

Proof. We first observe that

$$S\text{-local morphisms} = L^{-1}(\mathrm{Isos})$$

since we have

$$\mathscr{C}(f,c) = \mathscr{C}(f,\iota c) \cong \mathscr{C}_S(Lf,c)$$

for every morphism f in $\mathscr{C}$ and every S-local object $c \in \mathscr{C}$. If

$$\mathscr{C} \underset{\mathfrak{U}}{\overset{\mathfrak{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathscr{D}$$

is a pair of adjoint functors such that the left adjoint $\mathfrak{F}$ inverts the morphisms of S. From the isomorphism

$$\mathscr{D}(\mathfrak{F}c, d) \cong \mathscr{C}(c, \mathfrak{U}d)$$

it then follows immediately that $\mathfrak{U}$ takes values in $\mathscr{C}_S$. This in turn, however, implies that $\mathfrak{F}$ inverts all S-local morphisms, and hence all morphisms that are inverted by L. Thus the claim follows from Proposition 6.16. □

So having all that, what is the relation with our notion of $\mathscr{J}$-sheaves for a site $(\mathscr{C},\mathscr{J})$? We may consider the fully faithful inclusion of $\mathscr{J}$-sheaves into presheaves on $\mathscr{C}$:

$$\mathrm{Sh}_{(\mathscr{C},\mathscr{J})} \xhookrightarrow{\ \iota\ } \mathrm{Psh}_{\mathscr{C}}$$

This functor has a left adjoint if and only if the right Kan extension

$$\begin{array}{ccc} \mathrm{Sh}_{(\mathscr{C},\mathscr{J})} & = & \mathrm{Sh}_{(\mathscr{C},\mathscr{J})} \\ {\scriptstyle\iota}\big\downarrow & \nearrow_{L:=\mathrm{Ran}_\iota 1} & \\ \mathrm{Psh}_{\mathscr{C}} & & \end{array}$$

exists and is preserved by ι (this follows from Theorem 3.10). The category $\mathrm{Sh}_{(\mathscr{C},\mathscr{J})}$ is complete however, since limits commute with limits (the defining condition of a sheaf involves an equalizer diagram) and we may compute limits of sheaves in the category of presheaves. Therefore, $L = \mathrm{Ran}_\iota 1$ exists and, moreover, it is preserved by ι (ι leaves the Kan extension untouched). This implies, along with Proposition 6.16, the following result:

Theorem 6.19. *Every category of sheaves is a reflective subcategory*

$$\mathrm{Sh}_{(\mathscr{C},\mathscr{J})} \underset{\iota}{\overset{L}{\rightleftarrows}} \mathrm{Psh}_{\mathscr{C}} \quad (L \dashv \iota)$$

In particular, the category of sheaves is equivalent to the localization $\mathrm{Psh}_{\mathscr{C}}[\mathscr{W}^{-1}]$ *with* $\mathscr{W} := L^{-1}(\mathrm{Isos})$.

Remark 6.20. In fact, even more than the previous Theorem holds true: First of all one may prove that the reflector $L\colon \mathrm{Psh}_{\mathscr{C}} \to \mathrm{Sh}_{(\mathscr{C},\mathscr{J})}$ (also called *sheafification*) is left exact (preserves finite limits). Moreover, every *Grothendieck topos* arises in this way: Given a small category $\mathscr{C}$ there is a bijection between

- the equivalence classes of left exact reflective subcategories $\mathscr{E} \hookrightarrow \mathrm{Psh}_{\mathscr{C}}$ of the category of presheaves
- Grothendieck topologies $\mathscr{J}$ on $\mathscr{C}$,

which are such that $\mathscr{E} \approx \mathrm{Sh}_{(\mathscr{C},\mathscr{J})}$. See the Nlab link sheaf toposes are equivalently the left exact reflective subcategories of presheaf toposes.

6.3. **(Left) Bousfield Localization.** This chapter is based on the Nlab article Bousfield localization of model categories.

Bousfield Localization is, very roughly speaking, a procedure that takes a model category as input and spits out a new model category with more weak equivalences.

$$\mathscr{C} \xrightarrow{\text{left Bousfield localization}} \mathscr{C}_{\mathrm{loc}}$$

Bousfield localization will be a most crucial tool in the later chapters on ∞-categories. The main idea will be to take a certain model category of simplicial presheaves - e.g. $\mathrm{Psh}_\Delta(\Delta^{\times d})_{\mathrm{inj}}$ - on which we will perform a left Bousfield localization so as to single out ∞-categories as fibrant objects in the new model category structure. Therefore, it is helpful to think of Bousfield localization as the procedure which singles out certain kinds of objects (the fibrant objects in the new model structure) and provides these with a new ambient homotopy theory of sorts.

Definition 6.21. A *left Bousfield localization* $\mathscr{C}_{\mathrm{loc}}$ of a model category $\mathscr{C}$ is another model category structure on the underlying category $\mathscr{C}$ with the same cofibrations

$$\mathrm{Cof}_{\mathscr{C}_{\mathrm{loc}}} = \mathrm{Cof}_{\mathscr{C}}$$

but with more weak equivalences

$$\mathscr{W}_{\mathscr{C}_{\mathrm{loc}}} \supset \mathscr{W}_{\mathscr{C}}$$

Remark 6.22. It follows directly from the definition of a left Bousfield localization that

- $\mathrm{Fib}_{\mathscr{C}_{\mathrm{loc}}} = \left(\mathrm{Cof}_{\mathscr{C}_{\mathrm{loc}}} \cap \mathscr{W}_{\mathscr{C}_{\mathrm{loc}}}\right)^{\boxslash} \subset \left(\mathrm{Cof}_{\mathscr{C}_{\mathrm{loc}}} \cap \mathscr{W}_{\mathscr{C}}\right)^{\boxslash} = \mathrm{Fib}_{\mathscr{C}}$
- $\mathrm{Fib}_{\mathscr{C}_{\mathrm{loc}}} \cap \mathscr{W}_{\mathscr{C}_{\mathrm{loc}}} = \mathrm{Cof}^{\boxslash}_{\mathscr{C}_{\mathrm{loc}}} = \mathrm{Cof}^{\boxslash}_{\mathscr{C}} = \mathrm{Fib}_{\mathscr{C}} \cap \mathscr{W}_{\mathscr{C}}$
- The identity functor $\mathscr{C} \to \mathscr{C}_{\mathrm{loc}}$ preserves cofibrations and weak equivalences.
- The identity functor $\mathscr{C}_{\mathrm{loc}} \to \mathscr{C}$ preserves fibrations and trivial fibrations.
- Consequently, the pair of identity functors constitutes a Quillen adjunction

$$\mathscr{C} \underset{\perp}{\rightleftarrows} \mathscr{C}_{\mathrm{loc}}$$

Definition 6.23. Let $\mathscr{C}$ and $\mathscr{D}$ be model categories, and let

$$\mathscr{C} \underset{\mathfrak{U}}{\overset{\mathfrak{F}}{\leftrightarrows}} \mathscr{D} \quad (\perp)$$

be a Quillen adjunction. Then this adjunction is called a *Quillen reflection* if the induced derived adjunction (see Theorem 5.70)

$$\mathrm{Ho}\mathscr{C} \underset{\mathbf{R}\mathfrak{U}}{\overset{\mathbf{L}\mathfrak{F}}{\leftrightarrows}} \mathrm{Ho}\mathscr{D} \quad (\perp)$$

is a reflective subcategory-inclusion.

Proposition 6.24. *Let $\mathscr{C}$ be a model category. Then any left Bousfield localization $\mathscr{C}_{\mathrm{loc}}$ of $\mathscr{C}$ constitutes a Quillen reflection. More precisely, a left Bousfield localization constitutes a Quillen adjunction between identity functors*

$$\mathscr{C}_{\mathrm{loc}} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\leftrightarrows}} \mathscr{C} \quad (\perp)$$

which is a Quillen reflection. In particular, the induced derived adjunction

$$\mathrm{Ho}\mathscr{C}_{\mathrm{loc}} \underset{\mathbf{R}\mathrm{id}}{\overset{\mathbf{L}\mathrm{id}}{\leftrightarrows}} \mathrm{Ho}\mathscr{C} \quad (\perp)$$

is a reflective subcategory-inclusion.

Proof. See Example 2.2 on the Nlab Quillen reflection. □

Remark 6.25. The idea of a Quillen reflection is that of a homotopical reflective subcategory-inclusion. Proposition 6.24 tells us that left Bousfield localization is a particularly nice such homotopical reflective subcategory inclusion.

Throughout, we shall assume that $\mathscr{C}$ is a simplicial model category. Let $S \subset \mathrm{Mor}\mathscr{C}$ be a subclass of morphisms. Recall that in an ordinary category $\mathscr{C}$, a morphism f in $\mathscr{C}$ is an isomorphism if and only if for all objects $x \in \mathscr{C}$ the morphism

$$\mathscr{C}(f, x) \colon \mathscr{C}(\mathrm{cod} f, x) \to \mathscr{C}(\mathrm{dom} f, x)$$

is an isomorphism. Guided by this fact we have the following definition (very much reminiscient of we did in the previous chapter):

Definition 6.26. Let $\mathscr{C}$ and S be as described above.

- An object $X \in \mathscr{C}$ is called *S-local object* if for all s in S the induced morphism on derived hom-spaces

$$\mathbb{R}\mathrm{Hom}(\mathrm{cod}(s), X) \xrightarrow{\mathbb{R}\mathrm{Hom}(s,X)} \mathbb{R}\mathrm{Hom}(\mathrm{dom}(s), X)$$

  is a weak equivalence of simplicial sets.
- A morphism f in $\mathscr{C}$ is called an *S-local weak equivalence* or *S-equivalence* if for all S-local objects $X \in \mathscr{C}$ the morphism

$$\mathbb{R}\mathrm{Hom}(\mathrm{cod} f, X) \xrightarrow{\mathbb{R}\mathrm{Hom}(f,X)} \mathbb{R}\mathrm{Hom}(\mathrm{dom} f, X)$$

  is a weak equivalence of simplicial sets.
- We write $\mathscr{W}_S$ for the collection of all *S-local weak equivalences.*

Remark 6.27. The above definitions may be rephrased by using

$$\mathbb{R}\mathrm{Hom}(X, Y) \simeq \mathscr{C}(LX, RY)$$

for $L, R \colon \mathscr{C} \to \mathscr{C}$ cofibrant and fibrant replacement functors and all $X, Y \in \mathscr{C}$.

Proposition 6.28. *Let $\mathscr{C}$ be a 'nice enough' simplicial model category.*

- *A fibrant object X is an S-local object if and only if for all $s \in S$ the morphism*

$$\mathscr{C}(\mathrm{cod}(s), X) \xrightarrow{\mathscr{C}(s,X)} \mathscr{C}(\mathrm{dom}(s), X)$$

  is a trivial Kan fibration.
- *A cofibration f in $\mathscr{C}$ is an S-local weak equivalence if for all S-local fibrant objects X the morphism*

$$\mathscr{C}(\mathrm{cod} f, X) \xrightarrow{\mathscr{C}(f,X)} \mathscr{C}(\mathrm{dom} f, X)$$

  is a trivial Kan fibration.

Remark 6.29. For what we mean by 'nice enough' we refer the reader to the corresponding nlab article Bousfield localization of model categories as well as the corresponding section on Bousfield localization in [16].

If $\mathscr{C}$ is a simplicial model category and f is a weak equivalence between cofibrant objects in $\mathscr{C}$, then it follows from the axioms that

$$\mathscr{C}(\mathrm{cod} f, X) \xrightarrow{\mathscr{C}(f,X)} \mathscr{C}(\mathrm{dom} f, X)$$

is a weak equivalence of simplicial sets for all fibrant objects X. In particular, since the cofibrant replacement functor $L \colon \mathscr{C} \to \mathscr{C}$ is homotopical by the 2-out-of-3 axiom we get:

Lemma 6.30. *Every ordinary weak equivalence in $\mathscr{C}$ is also an S-local weak equivalence:*

$$\mathscr{W} \subset \mathscr{W}_S$$

Definition 6.31. The *left Bousfield localization* $\mathrm{L}_S\mathscr{C}$ of a given model category $\mathscr{C}$ at a class of morphisms S is, if it exists, the new model category structure on $\mathscr{C}$ with

- $\mathrm{Cof}_{\mathrm{L}_S\mathscr{C}} = \mathrm{Cof}_{\mathscr{C}}$
- $\mathrm{Cof}_{\mathrm{L}_S\mathscr{C}} \cap \mathscr{W}_{\mathrm{L}_S\mathscr{C}} = \mathrm{Cof}_{\mathscr{C}} \cap \mathscr{W}_S$

Proposition 6.32. *Assuming the left Bousfield localization exists as above, fibrant objects in $\mathrm{L}_S\mathscr{C}$ are precisely the fibrant objects in $\mathscr{C}$ that are S-local objects.*

One now wonders if any left Bousfield localization is induced by a family of morphisms S in $\mathscr{C}$:

Proposition 6.33. *In the context of left proper, cofibrantly generated simplicial model categories (see the Nlab), for $\mathscr{C}_{\mathrm{loc}}$ a left Bousfield localization of $\mathscr{C}$ as defined in Definition 6.21, there is a set $S \subset \mathrm{Mor}\mathscr{C}$ such that*

$$\mathscr{C}_{\mathrm{loc}} = \mathrm{L}_S\mathscr{C}$$

Theorem 6.34. *If $\mathscr{C}$ is a nice enough simplicial model category and $S \subset \mathrm{Mor}\mathscr{C}$ is a small set of morphisms, then the left Bousfield localization $\mathrm{L}_S\mathscr{C}$ does exist. Moreover, it satisfies the following conditions:*

- *The fibrant objects of $L_S\mathscr{C}$ are precisely the S-local objects of $\mathscr{C}$ that are fibrant in $\mathscr{C}$.*
- *$\mathrm{L}_S\mathscr{C}$ is itself a left proper model category.*
- *$L_S\mathscr{C}$ is itself a simplicial model category.*

Remark 6.35. See Bousfield localization of model categories and the corresponding section in [16] for more details on the technicalities. For our purposes all left Bousfield localizations exist, so we really don't need to indulge ourselves into technicalities.

Example 6.36. The following model categories $\mathscr{C}$ are nice enough, so that Theorem 6.34 is applicable and therefore, for every set $S \subset \mathrm{Mor}\mathscr{C}$, the left Bousfield localization $\mathrm{L}_S\mathscr{C}$ exists:

- The category Top endowed with the standard Quillen model structure on topological spaces.
- The category sSet endowed with the standard Quillen model structure on simplicial sets.
- The functor model categories $\mathscr{C}^{\mathscr{D}}_{\mathrm{inj}}$ for any simplicially enriched small category $\mathscr{D}$ and $\mathscr{C}$ a nice enough category, e.g., $\mathscr{C} = \mathrm{sSet}$.

In the upcoming sections, whenever we mention Bousfield localization, we shall refer to [16] for a pointer to existence results.

Remark 6.37. Bousfield localization may actually be defined via a universal property: Suppose $\mathscr{C}$ is a model category and S is a set of morphisms in $\mathscr{C}$. The *left Bousfield localization* of $\mathscr{C}$ at S is a model category $\mathrm{L}_S\mathscr{C}$ together with a left Quillen functor $\mathfrak{F}\colon \mathscr{C} \to \mathrm{L}_S\mathscr{C}$ that satisfies the following universal property: composing with $\mathfrak{F}$ maps left Quillen functors $\mathrm{L}_S\mathscr{C} \to \mathscr{D}$ bijectively to left Quillen functors $\mathfrak{U}\colon \mathscr{C} \to \mathscr{D}$ such that the left derived functor of $\mathfrak{U}$ sends elements of S to weak equivalences in $\mathscr{D}$. In other words, the map

$$\mathrm{lQ}(\mathrm{L}_S\mathscr{C}, \mathscr{D}) \xrightarrow{\quad \mathfrak{F}^\star \quad} \mathrm{lQ}(\mathscr{C}, \mathscr{D})_{\mathbb{L}}$$

is a bijection, where the LHS denotes left Quillen functors $\mathrm{L}_S\mathscr{C} \to \mathscr{D}$, while the RHS denotes left Quillen functors whose left derived functors send S to weak equivalences in $\mathscr{D}$.

6.4. **∞-Sheaves.** We shall now generalize the notions of presheaves and sheaves to simplicial presheaves and simplicial sheaves. The main difference is that a simplicial presheaf on a site $(\mathscr{C}, \mathscr{J})$ will take values in spaces, or more precisely, in simplicial sets.

Definition 6.38. A simplicial presheaf on a site $(\mathscr{C}, \mathscr{J})$ is a functor $\mathfrak{S}\colon \mathscr{C}^{\mathrm{op}} \to \mathrm{sSet}$. The category of simplicial presheaves will be denoted by

$$\mathrm{Psh}_\Delta(\mathscr{C}) := \mathrm{Psh}_{\Delta\times\mathscr{C}} = \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$$

We may also define a generalized version of Čech sieves, which are now referred to as Ĉech nerves:

Definition 6.39. Let $(\mathscr{C}, \mathscr{J})$ be a site and fix a covering $\mathscr{U} = \{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$.

- The *Čech nerve* $\mathbf{C}\mathscr{U}$ of the covering $\mathscr{U}$ is the simplicial presheaf given by

$$\mathbf{C}\mathscr{U} := \int^{[k]\in\Delta} \Delta^k \odot \coprod_{i_0,\ldots,i_n \in I} \text{よ}(U_{(i_0,\ldots,i_k)})$$

  where $U_{(i_0,\ldots,i_k)} := U_{i_0} \times_U \ldots \times_U U_{i_k}$ is the iterated pullback and $\text{よ} : \mathscr{C} \to \mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$ is the Yoneda embedding. In more detail, $\mathbf{C}\mathscr{U}$ is the simplicial presheaf which has as its n-simplices

$$\mathbf{C}\mathscr{U}_n := \coprod_{i_0,\ldots,i_n \in I} \text{よ} U_{(i_0,\ldots,i_n)}$$

  The simplicial structure maps d_k and s_k are given by projecting out or doubling the k-th factor, respectively:

$$\begin{array}{ccc} \coprod_{i_0,\ldots,i_n \in I} \text{よ} U_{(i_0,\ldots,i_n)} & \overset{\exists! d_k}{\dashrightarrow} & \coprod_{i_0,\ldots,i_{n-1} \in I} \text{よ} U_{(i_0,\ldots,i_n)} \\ \uparrow & & \uparrow \\ \text{よ} U_{(i'_0,\ldots,i'_n)} & \longrightarrow & \text{よ} U_{(i'_0,\ldots,\widehat{i'_k},\ldots,i'_n)} \end{array}$$

$$\begin{array}{ccc} \coprod_{i_0,\ldots,i_n \in I} \text{よ} U_{(i_0,\ldots,i_n)} & \overset{\exists! s_k}{\dashrightarrow} & \coprod_{i_0,\ldots,i_{n+1} \in I} \text{よ} U_{(i_0,\ldots,i_n)} \\ \uparrow & & \uparrow \\ \text{よ} U_{(i'_0,\ldots,i'_n)} & \longrightarrow & \text{よ} U_{(i'_0,\ldots,i'_k,i'_k,\ldots,i'_n)} \end{array}$$

- There is a canonical map

$$\mathbf{C}\mathscr{U} \longrightarrow \text{よ} U$$

  induced by the universal property

$$\begin{array}{ccc} \coprod_{i_0,\ldots,i_n \in I} \text{よ} U_{(i_0,\ldots,i_n)} & \dashrightarrow & \text{よ} U \\ \uparrow & \nearrow & \\ \text{よ} U_{(i'_0,\ldots,i'_n)} & & \end{array}$$

With these notions in hand we may now define the concept of an ∞-sheaf:

Definition 6.40. Let $(\mathscr{C}, \mathscr{J})$ be a site and consider the category of simplicial presheaves $\mathrm{Psh}_\Delta(\mathscr{C})_{\mathrm{inj}}$ endowed with the injective model structure. An ∞-*sheaf* is a simplicial presheaf $\mathfrak{S} \in \mathrm{Psh}_\Delta(\mathscr{C})$ which is

- a fibrant object with respect to the injective model structure on $\mathrm{Psh}_\Delta(\mathscr{C})_{\mathrm{inj}}$,
- local with respect to the canonical morphisms $\mathbf{C}\mathscr{U} \to \text{よ} U$ for every covering $\mathscr{U}$ in $\mathscr{J}$, that is, all induced morphisms

$$\mathbb{R}\mathrm{Map}(\text{よ} U, \mathfrak{S}) \longrightarrow \mathbb{R}\mathrm{Map}(\mathbf{C}\mathscr{U}, \mathfrak{S})$$

  are weak equivalences of simplicial sets.

Theorem 6.41. *There exists a model category structure* $\mathrm{Psh}_\Delta(\mathscr{C})_{\check{\mathrm{C}}\mathrm{ech}}$*, called the Čech model structure, given by performing left Bousfield localization on* $\mathrm{Psh}_\Delta(\mathscr{C})_{\mathrm{inj}}$ *at the canonical morphisms* $\mathbf{C}\mathscr{U} \to よU$*. Fibrant objects in this model category are precisely* ∞*-sheaves.*

Proposition 6.42. *Let* $X \in \mathrm{Psh}_\Delta(\mathscr{C})_{\mathrm{inj}}$*, or* $X \in \mathrm{Psh}_\Delta(\mathscr{C})_{\mathrm{loc}}$ *for some left Bousfield localization of the injective model structure on simplicial presheaves, then* X *may be written as the homotopy limit (with respect to the associated model structure)*

$$X \simeq \mathrm{hocolim}([n] \mapsto \mathrm{const} X_n)$$

Proof. Follows along the same lines as the proof of Corollary 5.105. □

We recall that any presheaf may be written as a colimit over representables by Corollary 3.9. In particular, any object in $\mathrm{Psh}_\Delta(\mathscr{C})_{\mathrm{inj}}$ or in $\mathrm{Psh}_\Delta(\mathscr{C})_{\mathrm{loc}}$ for any left Bousfield localization, is cofibrant. Hence:

Corollary 6.43. *Any object* $X \in \mathrm{Psh}_\Delta(\mathscr{C})_{\mathrm{loc}}$ *for any left Bousfield localization of* $\mathrm{Psh}_\Delta(\mathscr{C})_{\mathrm{inj}}$ *may be written as a homotopy colimit over representables:*

$$X \simeq \mathrm{hocolim}\, よ(c, [n])$$

Proof. Using Ken Brown's Lemma 5.17 we have

$$X \cong \mathrm{colim}\, よ(c, [n]) \simeq \mathrm{colim} L(よ(c, [n])) \simeq \mathrm{hocolim}\, よ(c, [n])$$

where L is some cofibrant replacement functor. □

7. ∞-Categories

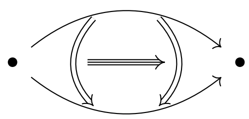
Bilbo: "Good morning!" said Bilbo, and he meant it. The sun was shining, and the grass was very green. But Gandalf looked at him from under long bushy eyebrows that stuck out further than the brim of his shady hat.
Gandalf: "What do you mean?" he said. "Do you wish me a good morning, or mean that it is a good morning whether I want it or not; or that you feel good this morning; or that it is a morning to be good on?"

J.R.R. Tolkien, The Hobbit

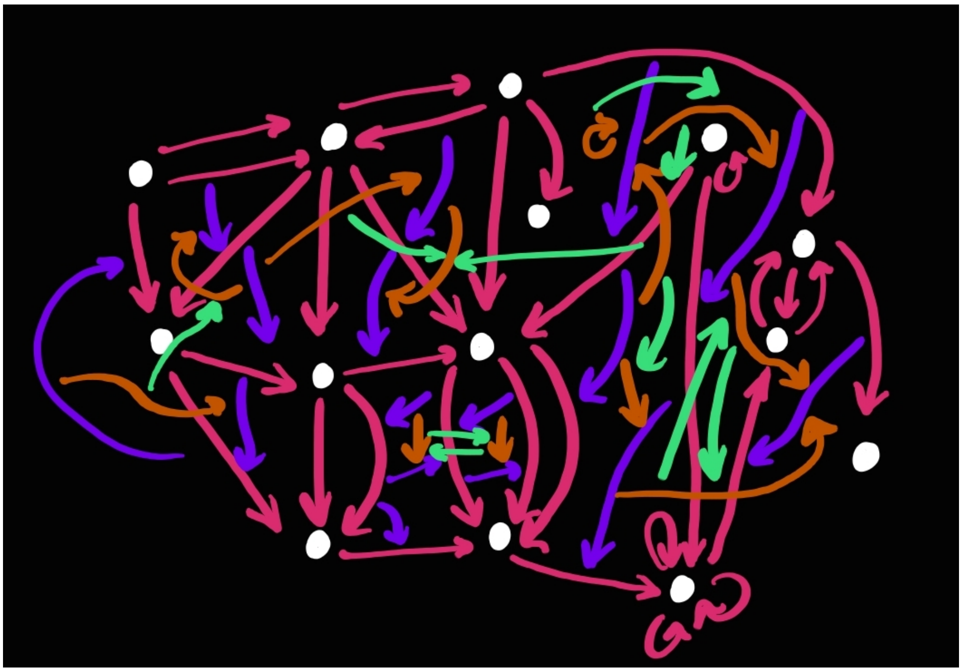
For the following chapters we will mostly follow [26], [24], [6], [31] and [30].

The notion of categories is a fundamental concept in mathematics that has found applications in many areas, including topology, algebraic geometry, and homotopy theory. However, in some cases, a single category is not enough to capture all the relevant information of a mathematical structure. This has led to the development of higher category theory and, more recently, ∞-category theory. For us it will be crucial to encode everything in terms of ∞-categories, as this will allow us to talk about *fully extended functorial field theories.*

7.1. **A Simplicial Perspective on Category Theory.** There are several approaches to define what it means to be an ∞-category. The idea is to generalize the standard notion of a category, which has objects and morphisms between those objects, to something which does not only have objects and morphisms, but also morphisms between morphisms, and then morphisms between morphisms of morphisms and so on. For example, the following picture visualizes two objects with two morphisms between these, and then an associated pair of 2-morphisms, along with a 3-morphism between the 2-morphisms:



More generally, the following picture shows a *multiple* 4-category which has objects (those being the white dots), morphisms between the objects (the pink arrows), and 2-morphisms between 1-morphisms (the purple arrows), 3-morphisms between 2-morphisms given by the orange arrows and 4-morphisms between the 3-morphisms (the green arrows). The word multiple highlights that higher morphisms, say a 2-morphism, doesn't have to have as domain and codomain 1-morphisms which both share the same domain and codomain themselves. A better name for this sort of category would probably be *quadruple category* (if one thinks of *double categories*). A 4-category which satisfies that its higher morphisms have as its source and target only morphisms which have the same domain and codomain will be called *globular* 4-*category* and the mentioned condition is usually referred to as *globularity.* Replacing 4 by an arbitrary d, we get a general sketch of definition for these things.



We mention that composition of (higher) morphisms should not be assumed to be unique in general, i.e., composition of morphisms should only be demanded up to homotopy (this will be discussed in the chapter on ∞-groupoids). Encoding both this property of higher morphisms and this notion of homotopy in the mentioned context leads quite naturally to a unison of the theory of simplicial sets (or more generally simplicial presheaves) and the theory of model categories, which in the end will result in a precise definition of (∞, d)-categories.

Before getting to all that it is important to realize that any definition of ∞-category should also include ordinary categories, i.e., any category should be an ∞-category where the higher morphisms are all simply identities. Therefore, in this chapter we will focus our resources towards showing that ordinary category theory may be encoded by means of simplicial sets, and simplicial sets in turn are encoded by ∞-categories (actually ∞-groupoids):

$$\text{Cat. Theory} \hookrightarrow \text{Simpl. Homotopy Theory} \hookrightarrow \infty\text{-Cat. Theory}$$

In order to get a feel for this, recall that for a category $\mathscr{C}$ we defined its nerve $\mathfrak{N}\mathscr{C}$ to be the simplicial set with n-simplices $\mathscr{C}^{[n]}$, where $[n] = \{0 \to 1 \to \ldots \to n\}$ is viewed as a category. This boils down to the following:

$$\mathfrak{N}\mathscr{C}_0 = \mathrm{Ob}\mathscr{C}$$
$$\mathfrak{N}\mathscr{C}_n = \mathscr{C}^{[1]} \times_{\mathscr{C}^{[0]}} \ldots \times_{\mathscr{C}^{[0]}} \mathscr{C}^{[1]}$$

where we recall that $\mathscr{C}^{[0]}$ corresponds to the set of objects of the category $\mathscr{C}$ and $\mathscr{C}^{[1]}$ corresponds to the set of arrows of $\mathscr{C}$. For example, if $n = 2$, consider the commutative diagram

$$\begin{array}{ccc} [2] & \xleftarrow{p_{1\to 2}} & [1] \\ {\scriptstyle p_{0\to 1}}\uparrow & & \uparrow{\scriptstyle p_0} \\ [1] & \xleftarrow[p_1]{} & [0] \end{array}$$

where the maps

(5) $$p_{i_0\to\ldots\to i_m}\colon [m] \to [n]$$

for $0 \leqslant i_0 \leqslant i_1 \leqslant \ldots \leqslant i_m \leqslant n$ are given by $[m] \ni j \mapsto i_j \in [n]$. Embedding this commutative diagram by means of the Yoneda Embedding yields a commutative square

$$\begin{array}{ccc} \Delta^2 & \longleftarrow & \Delta^1 \\ \uparrow & & \uparrow \\ \Delta^1 & \longleftarrow & \Delta^0 \end{array}$$

Since sSet is cocomplete the corresponding pushout $\Delta^1 \coprod_{\Delta^0} \Delta^1$ exists and therefore, by the universal property, there exists a unique map

$$\Delta^1 \textstyle\coprod_{\Delta^0} \Delta^1 \longrightarrow \Delta^2$$

The hom-functor $\mathrm{sSet}(-, \mathfrak{N}\mathscr{C})$ takes this map to an isomorphism:

$$\mathrm{sSet}(\Delta^2, \mathfrak{N}\mathscr{C}) \cong \mathscr{C}^{[2]} \xrightarrow{\ \cong\ } \mathscr{C}^{[1]} \times_{\mathscr{C}^{[0]}} \mathscr{C}^{[1]} \cong \mathrm{sSet}(\Delta^1 \textstyle\coprod_{\Delta^0} \Delta^1, \mathfrak{N}\mathscr{C})$$

Indeed, the diagram

$$\begin{array}{ccc} \mathscr{C}^{[2]} & \xrightarrow{p^\star_{1\to 2}} & \mathscr{C}^{[1]} \\ {\scriptstyle p^\star_{0\to 1}}\downarrow & & \downarrow{\scriptstyle p^\star_0} \\ \mathscr{C}^{[1]} & \xrightarrow[p^\star_1]{} & \mathscr{C}^{[0]} \end{array}$$

is a pullback square since $\mathscr{C}^{[2]}$ exactly agrees with

$$\begin{aligned} \mathscr{C}^{[1]} \times_{\mathscr{C}^{[0]}} \mathscr{C}^{[1]} &= \left\{(f, g) \in \mathscr{C}^{[1]} \times \mathscr{C}^{[1]} \mid p^\star_1 f = p^\star_0 g\right\} \\ &= \left\{(f, g) \in \mathscr{C}^{[1]} \times \mathscr{C}^{[1]} \mid \mathrm{cod} f = \mathrm{dom} g\right\} \end{aligned}$$

More generally, the commutative diagram

$$\begin{array}{ccc} [a+b] & \xleftarrow{p_{a\to\ldots\to a+b}} & [b] \\ {\scriptstyle p_{0\to\ldots\to a}}\uparrow & & \uparrow{\scriptstyle p_0} \\ [a] & \xleftarrow[p_a]{} & [0] \end{array}$$

induces an isomorphism

$$\mathscr{C}^{[a+b]} \xrightarrow{\ \cong\ } \mathscr{C}^{[a]} \times_{\mathscr{C}^{[0]}} \mathscr{C}^{[b]}$$

The property that $\mathfrak{N}\mathscr{C}$ induces the above isomorphisms may be phrased by saying that $\mathfrak{N}\mathscr{C}$ is *(strictly) local* with respect to the family of maps $\Delta^{a+b} \to \Delta^a \coprod_{\Delta^0} \Delta^b$. In particular, a functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ is completely encoded by $\mathfrak{N}\mathfrak{F}\colon \mathfrak{N}\mathscr{C} \to \mathfrak{N}\mathscr{D}$ in terms of the following data:

$$\begin{aligned} \mathfrak{N}\mathfrak{F}_0 &= \mathfrak{F}_{\mathscr{C}^{[0]}} \\ \mathfrak{N}\mathfrak{F}_n &= \mathfrak{F}_{\mathscr{C}^{[1]}} \times_{\mathscr{C}^{[0]}} \ldots \times_{\mathscr{C}^{[0]}} \mathfrak{F}_{\mathscr{C}^{[1]}} \end{aligned}$$

where $\mathfrak{F}_{\mathscr{C}^{[0]}} : \mathscr{C}^{[0]} \to \mathscr{D}^{[0]}$ and $\mathfrak{F}_{\mathscr{C}^{[1]}} : \mathscr{C}^{[1]} \to \mathscr{D}^{[1]}$ are the assignments induced by how the functor acts on objects and morphisms of $\mathscr{C}$. In this manner, we have seen that every category gives rise to a simplicial set and that the corresponding simplicial set is already fully determined if one knows about $\mathscr{C}^{[0]}$ and $\mathscr{C}^{[1]}$. Is the converse true?

Definition 7.1. A simplicial set X satisfies the *strict Segal condition* if the map

$$X_n \to X_1 \times_{X_0} \ldots \times_{X_0} X_1$$

induced by the universal property of the pullback

$$\begin{array}{ccc} X_a \times_{X_0} X_b & \dashrightarrow & X_b \\ \vdots & & \downarrow {\scriptstyle X_{p_0}} \\ X_a & \xrightarrow[X_{p_a}]{} & X_0 \end{array}$$

is a bijection for all $n \geqslant 2$.

Remark 7.2. Note that for a category $\mathscr{C}$ the assignment of domains and codomains of morphisms may be equivalently described in terms of the simplicial set $\mathfrak{N}\mathscr{C}$. Indeed, $\mathrm{dom}_{\mathscr{C}} = p_0^\star$ and $\mathrm{cod}_{\mathscr{C}} = p_1^\star$. Moreover, the codegeneracy map $s^0\colon [1] \to [0]$ and the degeneracy map $d^1\colon [1] \to [2]$ are equal to $p_{0\to 0}$ and $p_{0\to 2}$, respectively. Applying the simplicial set $\mathfrak{N}\mathscr{C}$ to s^0 and d^1 above yields the respective identity as well as composition morphisms.

Guided by the previous remark and the strict Segal condition we have the following:

Theorem 7.3. *A simplicial set X satisfies the strict Segal condition if and only if there exists a category $\mathscr{C}$ such that $\mathfrak{N}\mathscr{C}$ is naturally isomorphic to X.*

Proof. Define the category $\mathscr{C}$ as follows: The objects of $\mathscr{C}$ are given by the elements of the set X_0 and morphisms between these objects are defined by setting $\mathrm{Mor}\mathscr{C} = X_1$. Then source, target and identity maps are defined as $\mathrm{dom}_{\mathscr{C}} = d_1\colon X_1 \to X_0, \mathrm{cod}_{\mathscr{C}} = d_0\colon X_1 \to X_0, 1_{\mathscr{C}} = s_0\colon X_0 \to X_1$ and composition is given by $d_1\colon X_2 \cong X_1 \times_{X_0} X_1 \to X_1$. □

Proposition 7.4. *The nerve functor $\mathfrak{N}\colon \mathrm{Cat} \to \mathrm{sSet}$ is fully faithful.*

Proof. We have to prove that the map

$$\mathrm{Cat}(\mathscr{C}, \mathscr{D}) \to \mathrm{sSet}(\mathfrak{N}\mathscr{C}, \mathfrak{N}\mathscr{D})$$

has an inverse. Let $\zeta\colon \mathfrak{N}\mathscr{C} \to \mathfrak{N}\mathscr{D}$ be a natural transformation. Define $\psi(\zeta)$ as the functor $\mathscr{C} \to \mathscr{D}$ which on objects is equal to ζ_0 and on morphisms is equal to ζ_1. Simplicial identities verify that this is indeed a functor. It is then clear that for any functor $\mathfrak{F}\colon \mathscr{C} \to \mathscr{D}$ we have $\psi(\mathfrak{N}\mathfrak{F}) = \mathfrak{F}$, and on the other hand, $\mathfrak{N}\psi(\zeta) = \zeta$ as those maps agree at level 0 and level 1 and that already completely determines the map. □

The strict Segal conditions, or rather the structure of what it means for a simplicial set to actually be a standard category may also be encoded in terms of (inner) horn filling conditions. Recall the i-th horn Λ_i^n from Example 2.9. We may then also describe what happens to groupoids after having been embedded by the nerve functor:

Theorem 7.5. *Let $X \in \mathrm{sSet}$.*

- *X is the nerve of a category precisely if all inner horns have unique fillers: For all $0 < i < n$ any diagram of the form*

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & X \\ \big\downarrow & \overset{\exists !}{\nearrow} & \\ \Delta^n & & \end{array}$$

admits a unique lift $\Delta^n \to X$ making the diagram commute.
- *X is the nerve of a groupoid precisely if all horns have unique fillers: For all $0 \leqslant i \leqslant n$ any diagram of the form*

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & X \\ \big\downarrow & \overset{\exists !}{\nearrow} & \\ \Delta^n & & \end{array}$$

admits a unique lift $\Delta^n \to X$ making the diagram commute.

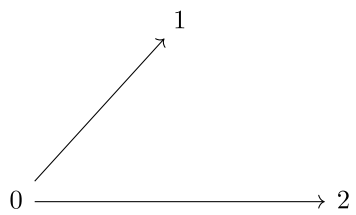
Let us make sense of why this might be true before getting to the actual proof of the result. Let $\{J_j\}_{j=1}^{n-1}$ be the subset of $\mathscr{P}([n])\backslash\{[n], \{0, \ldots, \widehat{i}, \ldots, n\}\}$ whose elements have cardinality $|J_i| = n$ for all i (there are exactly $n-1$ such sets). The horn Λ^n_i may then be identified with the iterated pushout

$$\Delta^{n,J_1} \coprod_{\Delta^{n,J_1 \cap J_2}} \Delta^{n,J_2} \coprod_{\Delta^{n,J_2 \cap J_3}} \cdots \coprod_{\Delta^{n,J_{n-2} \cap J_{n-1}}} \Delta^{n,J_{n-1}}$$

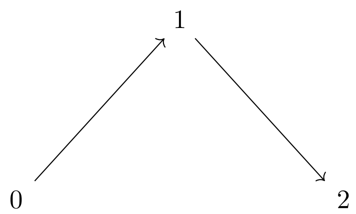
where $\Delta^{n,J}$ (with $J \in \{J_j\}_{j=1}^{n}$) is the corresponding simplicial subset of Δ^n with m-simplices

$$\Delta^{n,J}_m = \left\{ f \in \Delta^n_m \mid f([m]) \subset J \right\}$$

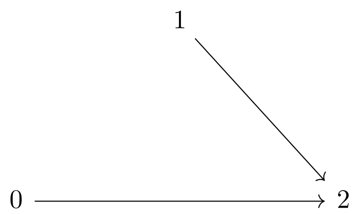
In particular, note that $\Delta^{n,J} \cong \Delta^{n-1}$. With that in mind, let us look at the case where $n = 2$. As we saw in Example 2.9, the horn Λ^2_0 may be depicted by



while Λ^2_1 looks like



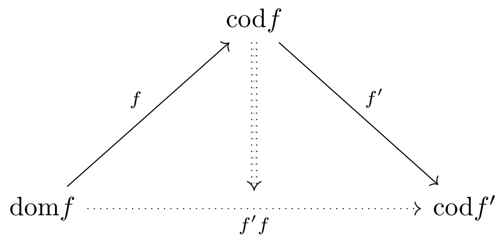
and Λ^2_2 looks like



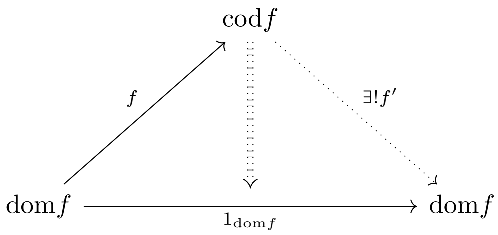
Since

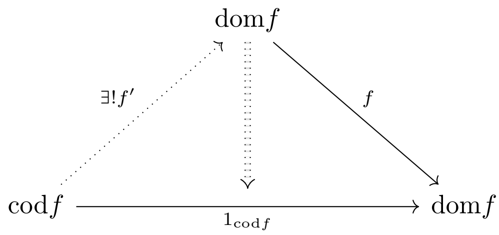
$$\begin{aligned} \mathrm{sSet}(\Lambda^2_1, X) &\cong \mathrm{sSet}\Big(\Delta^{2,\{0,1\}} \coprod_{\Delta^{2,\{1\}}} \Delta^{2,\{1,2\}},\ X\Big) \\ &\cong \mathrm{sSet}(\Delta^{2,\{0,1\}}, X) \prod_{\Delta^{2,\{1\}}} \mathrm{sSet}(\Delta^{2,\{1,2\}}, X) \\ &\cong \Big\{(f, f') \in X_1 \times X_1 \mid d_1 f = d_0 f'\Big\} \end{aligned}$$

any map $\Lambda^2_1 \to X$ corresponds to a pair (f, f') of 1-simplices of X such that $d_0 f = d_1 f'$ (one should think of composable morphisms). The (unique) lifting condition corresponding to Λ^2_1, which is the only *inner horn* for Δ^2, encapsulates the notion of X having unique composition in the sense that any two 1-simplices, for which the source is the target of the other, can be filled to a unique 2-simplex. The additional face is then thought of as the composite of the original two 1-simplices:



The *outer horns* encode something else entirely, however. The existence and uniqueness of lifts when $i = 0$ and $i = 2$ guarantee the existence of unique left and right inverses to a given 1-simplex:





Proof of Theorem 7.5. Let X be a simplicial set such that every inner horn has a unique filler. We will show that there is a category $\mathscr{C}$ and an isomorphism of simplicial sets $X \to \mathfrak{N}\mathscr{C}$:

- The objects of $\mathscr{C}$ are the vertices of X, i.e., $\mathrm{Ob}\mathscr{C} := X_0$.
- Given a pair of objects $c, c' \in \mathscr{C}$ the hom-set $\mathscr{C}(c, c')$ is defined as the set of 1-simplices $f \in X_1$ such that $d_1 f = c$ and $d_0 f = c'$.
- For each object $c \in \mathscr{C}$, we define the identity morphism $1_c \in \mathscr{C}(c, c)$ to be the 1-simplex $s_0(c)$.
- For objects $c, c', c'' \in \mathscr{C}$ and a pair of morphisms $f \in \mathscr{C}(c, c')$ and $f' \in \mathscr{C}(c', c'')$ we may apply the inner horn filling hypothesis to conclude that there is a unique 2-simplex $\sigma \in X_2$ satisfying $d_2\sigma = f$ and $d_0\sigma = f'$. We may then define the composition $f'f \in \mathscr{C}(c, c'')$ to be the edge $d_1\sigma$.

We then claim that $\mathscr{C}$ is a category. In order to check this we have to verify that the composition law is unital and associative. For unitality we have to prove:

$$1_{c'} f = f = f 1_c$$

In order to see the left identity, we must construct a 2-simplex $\sigma \in X_2$ so that $d_0\sigma = 1_{c'}$ and $d_1\sigma = d_2\sigma = f$. The degenerate 2-simplex $s_1 f$ has these properties.

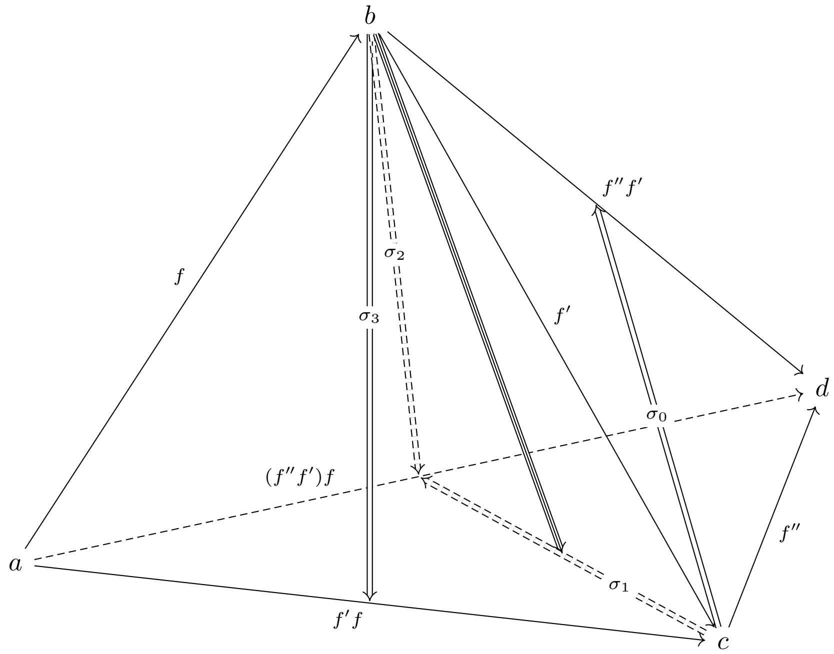
Let us check associativity now: For composable morphisms

$$a \xrightarrow{f} b \xrightarrow{f'} c \xrightarrow{f''} d$$

in $\mathscr{C}$ we have to verify $f''(f'f) = (f''f')f$. By repeatedly applying the inner horn filling property we deduce the following:

- There is a unique 2-simplex $\sigma_0 \in X_2$ satisfying $d_0\sigma_0 = f''$ and $d_2\sigma_0 = f'$.
- There is a unique 2-simplex $\sigma_3 \in X_2$ satisfying $d_0\sigma_3 = f'$ and $d_2\sigma_3 = f$.
- There is a unique 2-simplex $\sigma_2 \in X_2$ satisfying $d_0\sigma_2 = f''f'$ and $d_2\sigma_2 = f$.
- There is a unique 3-simplex $\eta \in X_3$ satisfying $d_0\eta = \sigma_0, d_2\eta = \sigma_2$ and $d_3\eta = \sigma_3$.

The 3-simplex yields a diagram



Setting $\sigma_1 := d_1\eta$ yields a 2-simplex which satisfies $d_0\sigma_1 = f'', d_1\sigma_1 = (f''f')f$ and $d_2\sigma_1 = f'f$. Hence σ_1 witnesses the identity $f''(f'f) = (f''f')f$. Finally, note that every n-simplex $\sigma\colon \Delta^n \to X$ determines a functor $[n] \to \mathscr{C}$, given on objects by values of σ on the set of vertices Δ^n_0 and on morphisms by the values of σ on the set of edges Δ^n_1. This determines a simplicial map $X \to \mathfrak{N}\mathscr{C}$, which is bijective on simplices of dimension $\leqslant 1$. For the remaining claims see [26] Lemma 1.2.3.2, Proposition 1.2.4.2. and Proposition 1.2.3.1. $\square$

So we have encoded categories and functors in the setting of simplicial sets. What about natural transformations then? Let $\mathscr{C}, \mathscr{D}$ be categories and view $[1] = \{0 \to 1\}$ as a category. We then recall that a natural transformation between two functors $\mathscr{C} \to \mathscr{D}$ is nothing else than a functor $\mathscr{C} \times [1] \to \mathscr{D}$. This is easily seen by making use of the internal hom adjunction

$$\mathrm{Cat}(\mathscr{C} \times [1], \mathscr{D}) \cong \mathrm{Cat}([1], [\mathscr{C}, \mathscr{D}])$$

and by realizing that functors with domain $[1]$ just pick out an arrow in the target category. In that spirit we see that natural transformations are equivalently morphisms $\mathfrak{N}\mathscr{C} \times \Delta^1 \to \mathfrak{N}\mathscr{D}$ since, first of all $\mathfrak{N}[1] = \Delta^1$ and therefore

$$\mathrm{Map}(\mathfrak{N}\mathscr{C}, \mathfrak{N}\mathscr{D})_1 := \mathrm{sSet}(\mathfrak{N}(\mathscr{C} \times [1]), \mathfrak{N}\mathscr{D}) \cong \mathrm{Cat}(\mathscr{C} \times [1], \mathscr{D})$$

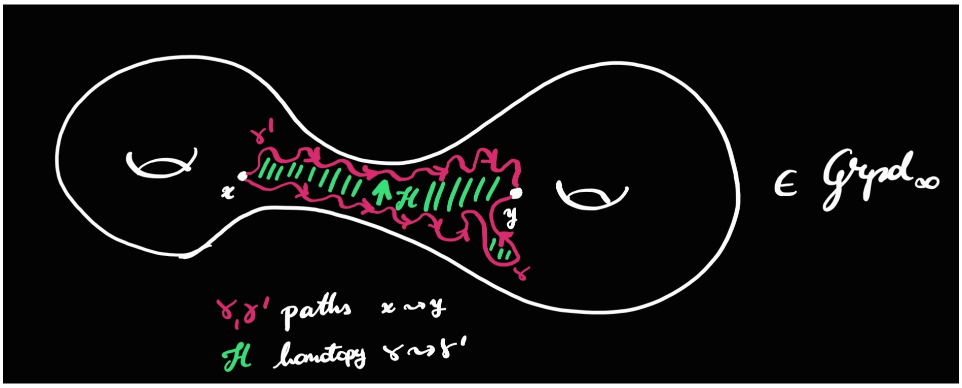
Summarizing all this amounts to the following: Category theory can be done just as well by means of only looking at simplicial sets, categories are encoded by their nerves, as are the corresponding functors, while natural transformations correspond to simplicial homotopies between the respective nerves of categories.

7.2. $(\infty, 0)$**-Categories aka** ∞**-Groupoids.** We have seen what really constitutes a category and how it can be (fully faithfully so) encoded as a simplicial set. We have also seen that, in this simplicial setting, a groupoid exactly corresponds to a strict Kan complex, i.e., all fillers are unique. The first notion of an ∞-category that we shall define (rigorously) is that of an ∞-*groupoid*. Very roughly speaking, an ∞-groupoid, like any ∞-category, has objects and k-morphisms for every natural number $k \geqslant 1$. However, an ∞-groupoid has, by definition, only invertible k-morphisms for all k. Recall that any topological space X may be viewed as a Kan complex $\Pi_{\leqslant\infty}X := \mathrm{Top}(|-|, X)$ (this is Theorem 2.43) which takes $[n]$ and maps it to the set of continuous maps $|\Delta^n| \to X$.

Example 7.6. Any topological space X gives rise to an ∞-groupoid:

- Objects are the points in X.
- 1-morphisms are paths from one point to the other.
- 2-morphisms are homotopies between paths.
- 3-morphisms are homotopies between homotopies between paths.

  ⋮
- k-morphisms are given by homotopies between homotopies between ... between paths.
- Composition is given by concatenation of paths, homotopies, etc. We note that this kind of composition is only unique up to (higher) homotopy. Identities are the constant paths, homotopies, etc.



We note that the above ∞-groupoid is readily encoded by the Kan complex $\Pi_{\leqslant\infty}X$. In particular, we realize that $\Pi_{\leqslant\infty}X$ only depends on the homotopy type of X.

With the above example in mind, *Grothendieck's Homotopy Hypothesis* argues that any sensible notion of ∞-category should imply that the collection of ∞-groupoids is already fully determined by taking homotopy types of topological spaces. In fact, Grothendieck's homotopy hypothesis states the following:

$$\text{Homotopy Types of Top. Spaces} \cong \infty\text{-Groupoids}$$

This motivates the following definition:

Definition 7.7. An $(\infty, 0)$-*category*, or just ∞-*groupoid*, is a Kan complex.

Remark 7.8. The prefix $(\infty, 0)$ should emphasize that there are infinitely many layers of higher morphisms, yet all of them are trivial in that they are isomorphisms (up to homotopy). More generally, an (∞, d)-category will mean an ∞-category which allows only the first d-layers of higher morphisms to be non-trivial, while all higher morphisms, starting with $(d+1)$-morphisms, will always be isomorphisms.

Example 7.9. Consider the real numbers $\mathbb{R}$ with its group structure induced by addition. We may view $\mathbb{R}$ as a category with a single object, while its set of morphisms is given by the real numbers themselves. Taking the nerve of this category, and again denoting it by $\mathbb{R}$, results in a Kan complex and hence an ∞-groupoid. This construction works for any Lie group.

With this definition one may verify the validity of the homotopy hypothesis, which is then nothing more than a restatement of Theorem 5.39 by means of Proposition 5.25:

Theorem 7.10 (see [15]). *Consider the full subcategories of bifibrant objects* $\mathrm{Kan}_{\mathrm{Quillen}} \hookrightarrow \mathrm{sSet}_{\mathrm{Quillen}}$ *and* $\mathrm{Spaces}_{\mathrm{Quillen}} \hookrightarrow \mathrm{Top}_{\mathrm{Quillen}}$. *Then the adjunction*

$$\mathrm{Kan}_{\mathrm{Quillen}} \underset{\Pi_{\leqslant\infty}}{\overset{|-|}{\rightleftarrows}} \mathrm{Spaces}_{\mathrm{Quillen}} \quad (\perp\mathrm{Quillen})$$

is a Quillen equivalence. In other words, the induced adjunction

$$\mathrm{Ho}(\mathrm{Kan}_{\mathrm{Quillen}}) \underset{\mathbf{R}\Pi_{\leqslant\infty}}{\overset{\mathbf{L}|-|}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Spaces}_{\mathrm{Quillen}}) \quad (\perp)$$

is an equivalence of categories.

Remark 7.11. This motivates why we would call a simplicial set a space.

What about ∞-*functors* and ∞-*natural transformations* between ∞-groupoids (Kan complexes)?

Definition 7.12. Let $\mathscr{C}, \mathscr{D}$ be ∞-groupoids, i.e., Kan complexes.

- An ∞-*functor* $\mathscr{C} \to \mathscr{D}$ is simply a natural transformation of the underlying Kan complexes.
- An ∞-*natural transformation* is a simplicial homotopy between two simplicial maps $\mathscr{C} \to \mathscr{D}$, that is, it is a map $\Delta^1 \times \mathscr{C} \to \mathscr{D}$.

We recall that the nerve functor $\mathfrak{N}\colon \mathrm{Cat} \to \mathrm{sSet}$ has a left adjoint $\mathfrak{h}\colon \mathrm{sSet} \to \mathrm{Cat}$, which maps a simplicial set to its associated homotopy category.

Remark 7.13 (see also [18]). Any ∞-functor between ∞-groupoids induces a functor between the respective homotopy categories (by applying $\mathfrak{h}$). Let us understand how an ∞-natural transformation $h\colon \mathscr{C} \times \Delta^1 \to \mathscr{D}$ induces a natural transformation between functors on the respective homotopy categories. First of all, the domain and codomain ∞-functors of h are given by

$$\begin{array}{ccccc}
\mathscr{C} \times \Delta^0 & \xrightarrow{1\times d^1} & \mathscr{C} \times \Delta^1 & \xleftarrow{1\times d^0} & \mathscr{C} \times \Delta^0 \\
\uparrow \cong & & \downarrow h & & \uparrow \cong \\
\mathscr{C} & \underset{\zeta_1 := h(1\times d^1)}{\dashrightarrow} & \mathscr{D} & \underset{\zeta_2 := h(1\times d^0)}{\dashleftarrow} & \mathscr{C}
\end{array}$$

Next note that any 1-morphism $f \in \mathscr{C}_1$ may be interpreted as a map $\Delta^1 \xrightarrow{f} \mathscr{C}$ (by the Yoneda Lemma). Hence any such f induces a morphism h_f given by the composition

$$\Delta^1 \times \Delta^1 \xrightarrow{f\times 1} \mathscr{C} \times \Delta^1 \xrightarrow{h} \mathscr{D}$$

This in turn results in a square of 1-morphisms in $\mathscr{D}$ by considering the four maps:

$$\Delta^1 \times \Delta^0 \underset{1\times d^0}{\overset{1\times d^1}{\rightrightarrows}} \Delta^1 \times \Delta^1 \xrightarrow{f\times 1} \mathscr{C} \times \Delta^1 \xrightarrow{h} \mathscr{D}$$

$$\Delta^0 \times \Delta^1 \underset{d^0\times 1}{\overset{d^1\times 1}{\rightrightarrows}} \Delta^1 \times \Delta^1 \xrightarrow{f\times 1} \mathscr{C} \times \Delta^1 \xrightarrow{h} \mathscr{D}$$

These may be depicted by

$$\begin{array}{ccc} \zeta_1(\mathrm{dom} f) & \xrightarrow{h_{\mathrm{dom} f} := h_f(d^1\times 1) = h(d^1 \times f)} & \zeta_2(\mathrm{dom} f) \\ {\scriptstyle \zeta_1 f = h(1\times d^1)(f)}\downarrow & & \downarrow{\scriptstyle \zeta_2 f = h(1\times d^0)(f)} \\ \zeta_1(\mathrm{cod} f) & \xrightarrow[h_{\mathrm{cod} f} := h_f(d^0\times 1) = h(d^0 \times f)]{} & \zeta_2(\mathrm{cod} f) \end{array}$$

which already looks like the naturality square. What is left to show is that after passing to homotopy categories the above diagram commutes. We note that, in order to show this, it suffices to prove that the homotopy category $\mathfrak{h}(\Delta^1 \times \Delta^1)$ is the category obtained from the diagram

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

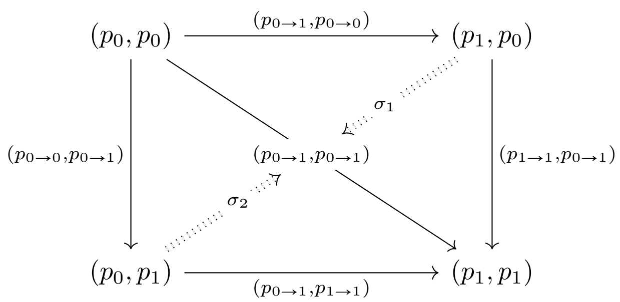
such that the two possible nontrivial compositions agree. This suffices because any morphism $\Delta^1 \times \Delta^1 \to \mathscr{D}$ induces a functor on the respective homotopy categories, which is then nothing else than a commutative square in $\mathfrak{h}\mathscr{D}$: The objects of $\mathfrak{h}(\Delta^1 \times \Delta^1)$ are the elements

$$\Delta^1_0 \times \Delta^1_0 = \{(p_0, p_0), (p_0, p_1), (p_1, p_0), (p_1, p_1)\}$$

where we use the notation of the maps defined in (5). Moreover, we have

$$\Delta^1_1 = \{p_{0\to 0}, p_{0\to 1}, p_{1\to 1}\}, \qquad \Delta^1_2 = \{p_{0\to 0\to 0}, p_{0\to 0\to 1}, p_{0\to 1\to 1}, p_{1\to 1\to 1}\}$$

The morphisms in $\mathfrak{h}(\Delta^1 \times \Delta^1)$ are given by equivalences classes of elements in $\Delta^1_1 \times \Delta^1_1$. We may depict most of these by



where

$$\sigma_1 := (p_{0\to 1\to 1}, p_{0\to 0\to 1}), \qquad \sigma_2 := (p_{0\to 0\to 1}, p_{0\to 1\to 1})$$

showing commutativity of the whole square in $\mathfrak{h}\mathscr{D}$. The remaining elements which we have not depicted above are all equivalent to some identity morphism on one of the four objects. This proves the claim.

7.3. $(\infty, 1)$**-Categories.** Grothendieck's homotopy hypothesis has guided us towards a sensible definition of ∞-groupoids. How to continue from here on out? Let us recall first that strict higher categories, say a strict 2-category, is simply a Cat-enriched category $\mathscr{C}$, i.e., $\mathscr{C}$ has a collection of objects and a category of 1-morphisms between these objects. This then also gives a recursive definition: A strict d-category is a category enriched over a strict $(d-1)$-category. We shall do something similar, which one would liken to something along the lines of weak enrichment (whatever that might mean precisely). In fact, an $(\infty, 1)$-category shall be a category *weakly enriched* in $(\infty, 0)$-categories. More generally, an (∞, d)-category will be a category *weakly enriched* in $(\infty, d-1)$-categories. More precisely, an $(\infty, 1)$-category will have a set of objects $\mathscr{C}_0$, and an ∞-groupoid (space) of 1-morphisms $\mathscr{C}_1$, which in turn has objects $\mathscr{C}_{1,0}$ which will yield 1-morphisms in $\mathscr{C}$, 1-morphisms $\mathscr{C}_{1,1}$ which constitute 2-morphisms in $\mathscr{C}$ and so on. Recursively, an (∞, d)-category will have a set of objects $\mathscr{C}_0$ and an $(\infty, d-1)$-category of 1-morphisms $\mathscr{C}_1$. To encode this rigorously, the first idea we might have is to add a higher categorical layer by adding another simplicial level:

Definition 7.14. The category of *simplicial spaces* is the category of simplicial presheaves $\mathrm{Psh}_\Delta(\Delta) = \mathrm{sSet}^{\Delta^{\mathrm{op}}}$.

Remark 7.15. In the literature, a simplicial space $X \in \mathrm{Psh}_\Delta(\Delta)$ is also sometimes called *bisimplicial set*. We will often, tacitly so, make use of the identifications

$$\mathrm{Psh}(\Delta^{\times 2}) \cong \mathrm{Psh}_\Delta(\Delta) \cong \mathrm{Set}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$$

where $\Delta^{\times 2} := \Delta \times \Delta$.

Notation 7.16. For $X \in \mathrm{sSet}$ we may want to emphasize that X has only really one slot where objects and morphisms can be plugged into. This is why we might be tempted to write $X = X_\bullet$. On the other hand, for a simplicial space $X \in \mathrm{Psh}_\Delta(\Delta)$ we have two such free slots, so we might want to write $X = X_{\bullet\bullet}$.

Notation 7.17. Instead of using

$$\mathrm{Psh}_\Delta(\Delta)(-,-)$$

as notation for the corresponding hom-set-functor, we shall simply write

$$\mathrm{Hom}(-,-)$$

if there is no danger of ambiguity. This will be more comfortable whenever we consider hom-sets between simplicial presheaves.

There are two canonical ways to turn a simplicial set $X \in \mathrm{sSet}$ into a bisimplicial set:

Definition 7.18. Let $\pi_i \colon \Delta^{\times 2} \to \Delta$ for $i = 1, 2$ be the corresponding projections on the first and second component.

- π_1 induces a functor $\pi_1^\star \colon \mathrm{sSet} \to \mathrm{Psh}_\Delta(\Delta)$ which takes a simplicial set X and maps it to the bisimplicial set $X_{\bullet\star}$ with bisimplices
$$(X_{\bullet\star})_{kl} := X_k$$
- π_2 induces a functor $\pi_2^\star \colon \mathrm{sSet} \to \mathrm{Psh}_\Delta(\Delta)$ which takes a simplicial set X and maps it to the bisimplicial set $X_{\star\bullet}$ with bisimplices
$$(X_{\star\bullet})_{kl} := X_l$$

Remark 7.19. The notation in the above definition is quite suggestive: $X_{\bullet\star}$ tells you that we view X as a bisimplicial set, but we really only have one slot where stuff can be inserted and this slot is the first factor in the product.

Remark 7.20. Let us point out some details:

- Looking more closely at Definition 7.18 we find that
$$\text{よ}_{\Delta^{\times 2}}([n],[m]) = \Delta^n_{\bullet\star} \times \Delta^m_{\star\bullet}$$
In particular, for any simplicial space $X \in \mathrm{Psh}(\Delta^{\times 2})$ we have
$$\mathrm{Hom}(\Delta^n_{\bullet\star} \times \Delta^m_{\star\bullet}, X) \cong X_{n,m}$$
by the Yoneda Lemma (recall that $\mathrm{Hom}(-,-)$ was introduced in Notation 7.17).
- From Example 4.14 we know that the category $\mathrm{Psh}_\Delta(\Delta)$ is cartesian closed. In particular, this yields that $\mathrm{Psh}_\Delta(\Delta)$ is enriched over simplicial sets by defining
$$\mathrm{Map}(X,Y) := \mathrm{Hom}(X \times \pi_2^\star \text{よ}_\Delta, Y)$$
for all simplicial spaces X, where $\pi_2 \colon \Delta^{\times 2} \to \Delta$ is again the projection onto the second factor. By the Yoneda Lemma we then have the following:
$$\mathrm{Map}(\Delta^n_{\bullet\star}, X) = \mathrm{Hom}(\Delta^n_{\bullet\star} \times \pi_2^\star \text{よ}_\Delta, Y) \cong Y_{n\bullet}$$

Recall that the injective model structure for $\mathrm{Psh}_\Delta(\Delta)$ exists and that all objects in $\mathrm{Psh}_\Delta(\Delta)_{\mathrm{inj}}$ are cofibrant.

Applying the Yoneda embedding よ_Δ to the commutative square

$$\begin{array}{ccc} [a+b] & \xleftarrow{p_{a\to\dots\to a+b}} & [b] \\ {\scriptstyle p_{0\to\dots\to a}}\uparrow & & \uparrow{\scriptstyle p_0} \\ [a] & \xleftarrow[p_a]{} & [0] \end{array}$$

yields a commutative square of simplicial maps

$$\begin{array}{ccc} \Delta^{a+b} & \longleftarrow & \Delta^b \\ \uparrow & & \uparrow \\ \Delta^a & \longleftarrow & \Delta^0 \end{array}$$

However, since sSet is cocomplete, the corresponding pushout $\Delta^a \coprod_{\Delta^0} \Delta^b$ exists and therefore, by the respective universal property, induces a map

$$\Delta^a \coprod_{\Delta^0} \Delta^b \longrightarrow \Delta^{a+b}$$

Furthermore, if $\pi_1 \colon \Delta^{\times 2} \to \Delta$ is the projection onto the first factor, then applying $\pi_1^\star$ to the above morphisms lets us obtain a map of simplicial spaces:

$$\Delta^a_{\bullet\star} \coprod_{\Delta^0_{\bullet\star}} \Delta^b_{\bullet\star} \longrightarrow \Delta^{a+b}_{\bullet\star}$$

Definition 7.21. A *Segal space* is a simplicial space $\mathscr{C} \colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ which satisfies the following conditions:

- *Fibrancy*: $\mathscr{C}$ is fibrant with respect to the injective model structure $\mathrm{Psh}_\Delta(\Delta)_{\mathrm{inj}}$.

- *Segal's special* Δ*-condition*: $\mathscr{C}$ is local with respect to the maps

$$\Delta^a_{\bullet\star} \coprod_{\Delta^0_{\bullet\star}} \Delta^b_{\bullet\star} \longrightarrow \Delta^{a+b}_{\bullet\star}$$

for all $a, b \in \mathbb{N}$.

Remark 7.22. Let us unravel the previous definition: As we have seen in the chapter on left Bousfield localizations $\mathscr{C}$ being fibrant with respect to the old model structure and then also demanding that $\mathscr{C}$ is local with respect to the given family of maps will force $\mathscr{C}$ into a fibrant object with respect to the new model structure obtained by Bousfield localization. The fibrancy condition could also be understood as the aforementioned weak enrichment: Fibrant objects in $\mathrm{Psh}_\Delta(\Delta)_{\mathrm{inj}}$ are all those $\mathscr{C} \in \mathrm{Psh}_\Delta(\Delta)$ such that the map $\mathscr{C} \to \star$ has the right lifting property with respect to all trivial cofibrations in $\mathrm{Psh}_\Delta(\Delta)_{\mathrm{inj}}$:

$$\begin{array}{ccc} \mathscr{D} & \longrightarrow & \mathscr{C} \\ {\scriptstyle\simeq}\big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow \\ \mathscr{E} & \longrightarrow & \star \end{array}$$

However, the existence of such a lift in particular implies the existence of a corresponding lift objectwise. But this then implies that a Segal space must in particular be a Kan complex at each simplicial level:

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \mathscr{C}_{m,\bullet} \\ \big\downarrow & \overset{\exists}{\nearrow} & \\ \Delta^n & & \end{array}$$

for all $m, n \in \mathbb{N}$ and $0 \leqslant i \leqslant n$.

On the other hand, Segal's special Δ-condition tells us at which set of maps we want to (left Bousfield) localize at. This condition demands that the induced maps

$$\mathrm{Map}(\Delta^{a+b}_{\bullet\star}, X) \longrightarrow \mathrm{Map}(\Delta^a_{\bullet\star} \coprod_{\Delta^0_{\bullet\star}} \Delta^b_{\bullet\star}, X)$$

are trivial Kan fibrations in the Quillen model structure $\mathrm{sSet}_{\mathrm{Quillen}}$. But then

$$\mathrm{Map}(\Delta^{a+b}_{\bullet\star}, \mathscr{C}) \cong \mathrm{Hom}(\Delta^{a+b}_{\bullet\star} \times \pi_2^\star \text{よ}_\Delta, \mathscr{C}) \cong \mathscr{C}_{a+b} \in \mathrm{sSet}$$

where we avoided, for practicality, to emphasize the free slot by $\mathscr{C}_{(a+b),\bullet}$. Analogously,

$$\mathrm{Map}(\Delta^a_{\bullet\star} \coprod_{\Delta^0_{\bullet\star}} \Delta^b_{\bullet\star}, \mathscr{C}) \cong \mathscr{C}_a \times_{\mathscr{C}_0} \mathscr{C}_b$$

Hence $\mathscr{C}$ satisfies Segal's special Δ-condition if and only if the morphisms

$$\mathscr{C}_{a+b} \xrightarrow{\mathscr{C}(p_{0\to\ldots\to a}) \times \mathscr{C}(p_{a\to\ldots\to a+b})} \mathscr{C}_a \times_{\mathscr{C}_0} \mathscr{C}_b$$

are trivial Kan fibrations. This gives the correct notion of an up-to-equivalence composition for our potential model of $(\infty, 1)$-categories, which is directly motivated by Definition 7.1.

Theorem 7.23. *There is a model structure on the category of simplicial spaces, which we denote by* SeSp*, such that all objects are cofibrant and the fibrant objects are precisely the Segal spaces. In fact,* SeSp *is given by the left Bousfield localization*

$$\mathrm{SeSp} = \mathrm{L}_S(\mathrm{Psh}_\Delta(\Delta)_{\mathrm{inj}})$$

where S is the family of morphisms

$$\Delta^a_{\bullet\star} \coprod_{\Delta^0_{\bullet\star}} \Delta^b_{\bullet\star} \longrightarrow \Delta^{a+b}_{\bullet\star}$$

for all $a, b \in \mathbb{N}$.

7.3.1. *Why are Segal spaces good candidates for $(\infty, 1)$-categories?* Motivated yet again by the construction of the nerve of a category, if $\mathscr{C}_{\bullet\bullet}$ is a Segal space, then we view the vertices of $\mathscr{C}_0$ as the set of objects. For $x, y \in \mathscr{C}_{0,0}$ we define

$$\mathscr{C}(x, y) := \Delta^0 \times^x_{\mathscr{C}_0} \mathscr{C}_1 \times^y_{\mathscr{C}_0} \Delta^0$$

where the pullback is taken over the diagram

$$\Delta^0 \xrightarrow{x} \mathscr{C}_0 \xleftarrow{\text{dom}} \mathscr{C}_1 \xrightarrow{\text{cod}} \mathscr{C}_0 \xleftarrow{y} \Delta^0$$

with dom $:= \mathscr{C}_{d^1,\bullet}$ and cod $:= \mathscr{C}_{d^0,\bullet}$. Note that $\mathscr{C}(x, y)$ is, in particular, a homotopy pullback, since $\mathscr{C}_{1\bullet}$ is fibrant (a Kan complex). The Kan complex $\mathscr{C}(x, y)$ is viewed as the $(\infty, 0)$-category of 1-morphisms from x to y. More generally, $\mathscr{C}_n$ is viewed as the $(\infty, 0)$-category of n-tuples of composable morphisms together with a composition. The composition is the map $\mathscr{C}_n \to \mathscr{C}_1$ determined by the order-preserving function

$$p_{0\to n} \colon [1] \to [n], \qquad 0 < 1 \mapsto 0 < n$$

More precisely, $\mathscr{C}_{1,0}$ is the set of 1-morphisms of $\mathscr{C}_{\bullet\bullet}$ and by using the zig-zag of arrows

$$\mathscr{C}_1 \times_{\mathscr{C}_0} \ldots \times_{\mathscr{C}_0} \mathscr{C}_1 \xleftarrow{\ \simeq\ } \mathscr{C}_n \xrightarrow{\mathscr{C}(p_{0\to n})} \mathscr{C}_1$$

we are able to define a composition

$$\mathscr{C}_1 \times_{\mathscr{C}_0} \ldots \times_{\mathscr{C}_0} \mathscr{C}_1 \longrightarrow \mathscr{C}_1$$

which is unique up to homotopy (after all we pick an arbitrary weak inverse from the pullback into $\mathscr{C}_n$). Concretely, we consider the commutative diagram of simplicial sets

$$\begin{array}{ccc}
\varnothing & \longrightarrow & \mathscr{C}_n \\
\big\downarrow & \overset{\exists k}{\nearrow} & \big\downarrow \simeq \\
\Delta^0 & \xrightarrow[(f_1, \ldots, f_n)]{} & \mathscr{C}(x_0, \ldots, x_n)
\end{array}$$

where $\mathscr{C}(x_0, \ldots, x_n)$ is defined to be the iterated pullback

$$\Delta^0 \times^{x_0}_{\mathscr{C}_{0,\bullet}} \mathscr{C}_{1,\bullet} \times^{x_1}_{\mathscr{C}_{0,\bullet}} \ldots \times_{\mathscr{C}_{0,\bullet}} \mathscr{C}_{1,\bullet} \times^{x_n}_{\mathscr{C}_{0,\bullet}} \Delta^0$$

that results from the span

$$\Delta^0 \xrightarrow{x_0} \mathscr{C}_0 \xleftarrow{\text{dom}} \mathscr{C}_1 \xrightarrow{\text{cod}} \mathscr{C}_0 \xrightarrow{x_1} \ldots \xleftarrow{x_n} \Delta^0$$

The morphism $k \colon \Delta^0 \to \mathscr{C}_n$ exists, since the map to the right is a trivial fibration, and the map to the left is a cofibration in the Quillen model structure on sSet. Hence this diagram tells us that for any n-tuple

$$x_0 \xrightarrow{f_1} x_1 \longrightarrow \ldots \xrightarrow{f_n} x_n$$

(after all this is the same as a map $\Delta^0 \to \mathscr{C}(x_0, \ldots, x_n)$ by the Yoneda Lemma) there exists $k \in \mathscr{C}_{n,0}$ such that

$$\Big(\mathscr{C}(p_{0\to 1}) \times \ldots \times \mathscr{C}(p_{n-1\to n})\Big)(k) = (f_1, \ldots, f_n)$$

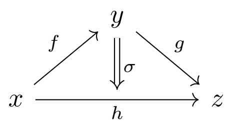
Composition is then defined by

$$f_n \dots f_1 := \mathscr{C}(p_{0\to n})(k)$$

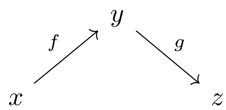
It is nice to think about these notions in a more geometric manner: Let us first inspect the condition that

$$\mathscr{C}_2 \xrightarrow{\simeq} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1$$

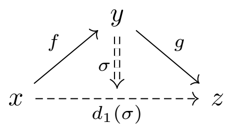
is a trivial Kan fibration. The domain of this map is the space of 2-cells σ which may be depicted by:



The codomain $\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1$, on the other hand, is the space of composable morphisms, which may be depicted by:



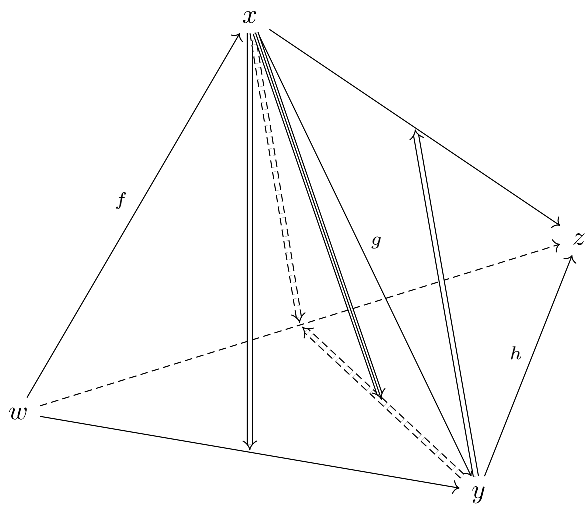
The Segal condition then says that every such composable pair of morphisms (f, g) can be filled out to a complete 2-cell:



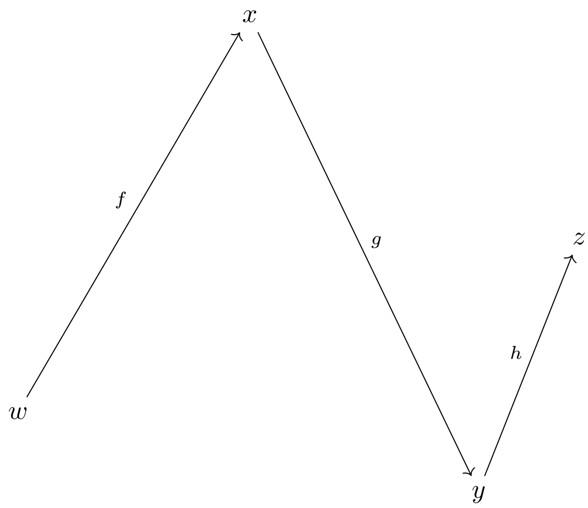
where $d_1 = \mathscr{C}_{d^1,0} = \mathscr{C}(p_{0\to 2})$. We think of $d_1(\sigma)$ as the composition of f and g, thus we will often just write gf. We notice that neither σ nor $d_1(\sigma)$ need to be unique here, however, we will see that such a composition is unique up to homotopy. The next condition is that

$$\mathscr{C}_3 \xrightarrow{\simeq} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1$$

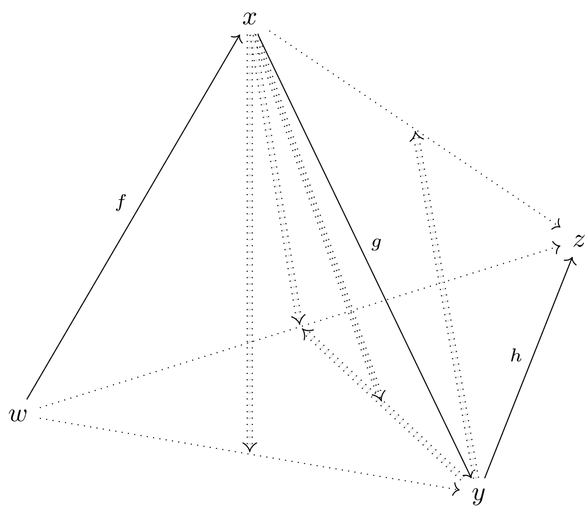
is a trivial Kan fibration. The domain of this map is the space of 3-cells which we can depict by a tetrahedron (or pyramid)



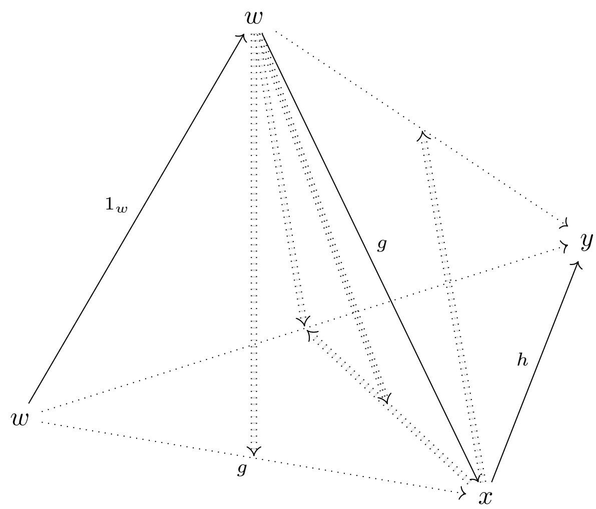
while the codomain $\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1$ may be depicted by a triple of arrows



The Segal condition implies that any such triplet of arrows may be filled out to give a complete 3-cell:



This assures that if we have three composable morphisms (f, g, h), then it doesn't really matter in which order we compose, that is, $(hg)f \sim h(gf)$, which is witnessed by the above 3-cell. Choosing $(f, g, h) = (1_{\mathrm{dom}(g)}, g, h)$, we in particular obtain that any two compositions of h and g are equivalent:



Thus, we have established how composition works for the space of 1-morphisms $\mathscr{C}_{1,\bullet}$. In particular, a 1-morphism or path in $\mathscr{C}_{1,\bullet}$ is an element in $\mathscr{C}_{1,1}$, i.e., a 2-morphism of $\mathscr{C}$ which is invertible up to weak equivalence. Composition of these 2-morphisms is achieved by means of the horn filling conditions of the Kan complex $\mathscr{C}_{1,\bullet}$. A 3-morphism in $\mathscr{C}$ is then simply and element in $\mathscr{C}_{1,2}$ and so on. In summary:

$\mathscr{C}_{0\bullet}$	...	space of objects of $\mathscr{C}$
$\mathscr{C}_{0,0}$	...	set of objects of $\mathscr{C}$
$\mathscr{C}_{1,\bullet}$	...	$(\infty, 0)$-category of 1-morphisms in $\mathscr{C}$
$\mathscr{C}_{n,\bullet}$	...	$(\infty, 0)$-category of n-tuples of composable arrows in $\mathscr{C}$
$\mathscr{C}_{1,0}$	...	set of 1-morphisms of $\mathscr{C}$
$\mathscr{C}_{1,k}$	...	set of $(k+1)$-morphisms of $\mathscr{C}$
$\mathscr{C}_{n,k}$	...	set of n-tuples of composable $(k+1)$-morphisms

Recall that $\pi_0 \colon \mathrm{Kan} \to \mathrm{Set}$ is the functor which takes a Kan complex $X \in \mathrm{Kan}$ and maps it to the coequalizer of the diagram

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0$$

Definition 7.24. The *homotopy category* $\mathfrak{h}_1(\mathscr{C})$ of a Segal space $\mathscr{C} = \mathscr{C}_{\bullet\bullet}$ is the (ordinary) category whose objects are given by the set $\mathscr{C}_{0,0}$ and whose morphisms between objects $x, y \in \mathscr{C}_{0,0}$ are

$$\mathfrak{h}_1(\mathscr{C})(x, y) := \pi_0 \mathscr{C}(x, y)$$

$$= \pi_0\left(\Delta^0 \times^x_{\mathscr{C}_0} \mathscr{C}_1 \times^y_{\mathscr{C}_0} \Delta^0\right)$$

For $x, y, z \in \mathscr{C}_{0,0}$ the following diagram induces composition of morphisms, as weak equivalences induce bijections on π_0:

$$\begin{array}{ccc}
\left(\Delta^0 \times^x_{\mathscr{C}_0} \mathscr{C}_1 \times^y_{\mathscr{C}_0} \Delta^0\right) \times \left(\Delta^0 \times^y_{\mathscr{C}_0} \mathscr{C}_1 \times^z_{\mathscr{C}_0} \Delta^0\right) & \dashrightarrow & \Delta^0 \times^x_{\mathscr{C}_0} \mathscr{C}_1 \times^z_{\mathscr{C}_0} \Delta^0 \\
\downarrow & & \uparrow {\scriptstyle \mathscr{C}(p_{0\to 2})} \\
\Delta^0 \times^x_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \times^z_{\mathscr{C}_0} \Delta^0 & \xleftarrow{\simeq} & \Delta^0 \times^x_{\mathscr{C}_0} \mathscr{C}_2 \times^z_{\mathscr{C}_0} \Delta^0
\end{array}$$

Lemma 7.25. *Let $\mathscr{C}$ be a model category and suppose that we are given a lifting problem*

$$\begin{array}{ccc}
A & \longrightarrow & C \\
{\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\
B & \longrightarrow & D
\end{array}$$

in $\mathscr{C}$, where i is a cofibration and p is a trivial fibration. Then any two lifts are left homotopic.

Proof. Suppose that we have two lifts $f, f'\colon B \to C$ for the above lifting problem. We then consider the commutative diagram

$$\begin{array}{ccccc}
B \coprod B & & \xrightarrow{f+f'} & & C \\
{\scriptstyle c_0+c_1}\downarrow & \searrow^{\nabla} & \nearrow^{\exists\zeta} & & \downarrow{\scriptstyle j} \\
\mathrm{Cyl}(B) & \xrightarrow[\simeq]{} & B & \longrightarrow & D
\end{array}$$

where $\mathrm{Cyl}(B)$ is a cylinder object for B and c_0, c_1 along with the weak equivalence $\mathrm{Cyl}(B) \to B$ constitute the corresponding extra structure. Since this diagram commutes by construction and the left vertical map is a cofibration, while the right vertical map is a trivial fibration, we obtain a lift $\zeta\colon \mathrm{Cyl}(B) \to C$, which establishes a left homotopy from f to f'. □

Corollary 7.26. *Let $\mathscr{C}$ be a Segal space and let $f, f' \in \mathscr{C}_{1,0}$ be composable* 1-*morphisms. Then any two choices for a composition of f and f' are homotopic, that is, there exists $F \in \mathscr{C}_{1,1}$ so that $\mathscr{C}_{1,d^1}(F) = f$ and $\mathscr{C}_{1,d^0}(F) = f'$. In particular, composition of* 1-*morphisms is a well defined map in the corresponding homotopy category.*

Proof. This follows immediately from the previous proposition, since taking a composite for f, f' boils down to the lifting problem

$$\begin{array}{ccc}
\varnothing & \longrightarrow & \mathscr{C}_2 \\
\downarrow & \nearrow & \downarrow{\scriptstyle \simeq} \\
\Delta^0 & \xrightarrow[(f,f')]{} & \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1
\end{array}$$

□

Proposition 7.27 ([31])**.** *For any Segal space $\mathscr{C} = \mathscr{C}_{\bullet\bullet}$ the homotopy category $\mathfrak{h}_1(\mathscr{C})$ is a category.*

Proof. In order to verify associativity we produce particular choices of compositions which give equal (not just homotopic) results. This is fully sufficient by the previous corollary. Let $k \in \mathscr{C}_{3,0}$ be such that

$$(\mathscr{C}p_{0\to 1} \times \mathscr{C}p_{1\to 2} \times \mathscr{C}p_{2\to 3})(k) = (f, g, h) \in \mathscr{C}(w, x, y, z)$$

Any such k determines compositions $gf := \mathscr{C}_{d^3,0}(k)$ and $h(gf) := \mathscr{C}_{d^1,0}(k)$, as well as $hg := \mathscr{C}_{d^0,0}(k)$ and $(hg)f := \mathscr{C}_{d^2,0}(k)$. Now let $\sigma := \mathscr{C}_{d^1,0}(k) \in \mathscr{C}_{2,0}$, then σ satisfies

$$\mathscr{C}_{d^0,0}(\sigma) = h, \qquad \mathscr{C}_{d^1,0}(\sigma) = (hg)f, \qquad \mathscr{C}_{d^2,0}(\sigma) = gf$$

In particular, σ witnesses the identity $h(gf) = (hg)f$, as desired. Identities in our category will be represented by $1_x := \mathscr{C}_{s^0,0}(x)$ for all $x \in \mathscr{C}_{0,0}$. To show that $f1_w \sim f$ for $f \in \mathscr{C}(w, x)$, let $k := \mathscr{C}_{s^0,0}(f)$. Then

$$(\mathscr{C}p_{0\to 1} \times \mathscr{C}p_{1\to 2})(k) = (1_w, f), \qquad \mathscr{C}_{d^1,0}(k) = f, \qquad \mathscr{C}_{d^0,0}(k) = f, \qquad \mathscr{C}_{d^1,0}(k) = 1_w$$

and therefore $f1_w = f$. The other identity follows analogously. □

Example 7.28. Let $\mathscr{C}$ be an (ordinary) small category. Let us view its corresponding nerve as the bisimplicial set $\mathfrak{N}\mathscr{C}_{\bullet\star}$. Then $\mathfrak{N}\mathscr{C}_{\bullet\star}$ is a Segal space and we have an equivalence of categories

$$\mathfrak{h}_1(\mathfrak{N}\mathscr{C}_{\bullet\star}) \approx \mathscr{C}$$

Definition 7.29. Let $\mathscr{C}$ be a Segal space.

- A 1-morphism $f \in \mathscr{C}(x, y)$ is called *invertible* if its image under

$$\mathscr{C}(x, y) \xrightarrow{\quad \pi_0 \quad} \pi_0\mathscr{C}(x, y)$$

  is an isomorphism.
- Two 1-morphisms $f, g \in \mathscr{C}(x, y)$ are called *homotopic*, if they lie in the same connected component of $\mathscr{C}(x, y)$, that is, if both these arrows represent the same equivalence class in $\pi_0\mathscr{C}(x, y)$. In that case, we write $f \sim g$

Unfortunately, Segal spaces do not quite provide the correct notion of $(\infty, 1)$-categories, albeit they provide a canonical composition that is unique up to weak equivalence. There are two reasons for this: The first of these reasons is that $\mathscr{C}_{0,\bullet}$ is a space rather than a set of objects. The second problem is that if we are given two Segal spaces $\mathscr{C}, \mathscr{D}$, the set of natural transformations $\mathscr{C} \to \mathscr{D}$ should be exactly the collection of ∞-functors from the ∞-category $\mathscr{C}$ to the ∞-category $\mathscr{D}$ (if we assume Segal spaces are the correct model for $(\infty, 1)$-categories). We would expect that the model structure on SeSp has as its set of weak equivalences exactly those ∞-functors which are fully faithful and essentially surjective (to be defined below) in a homotopical sense. In other words, we would want the weak equivalences of SeSp to be exactly equivalences between ∞-catgories.

Definition 7.30. Let $\mathscr{C}$ and $\mathscr{D}$ be Segal spaces. A natural transformation $\zeta\colon \mathscr{C} \to \mathscr{D}$ is called *Dwyer-Kan equivalence* if

- the induced map $\mathfrak{h}_1(\zeta)\colon \mathfrak{h}_1(\mathscr{C}) \to \mathfrak{h}_1(\mathscr{D})$ on homotopy categories is essentially surjective.
- for each pair of objects x, y in $\mathscr{C}$ the induced map $\mathscr{C}(x, y) \to \mathscr{D}(\zeta x, \zeta y)$ is a weak equivalence.

Remark 7.31. We realize that the map $\mathfrak{h}_1(\zeta)$ is well defined. Indeed, what we really need to check is that for any equivalence class $[f] \in \pi_0(\mathscr{C}_1)$ we have that $[\zeta_{1,0}(f)]$

is independent of the representative. But this follows from commutativity of the square

$$\begin{array}{ccc} \mathscr{C}_{1,0} & \xrightarrow{\zeta_{1,0}} & \mathscr{D}_{1,0} \\ \upuparrows & & \upuparrows \\ \mathscr{C}_{1,1} & \xrightarrow[\zeta_{1,1}]{} & \mathscr{D}_{1,1} \end{array}$$

where the upwards pointing vertical arrows are the maps $\mathscr{C}_{1,d^j}$ and $\mathscr{D}_{1,d^j}$ for $j = 0, 1$. Similarly, the induced map $\mathscr{C}(x, y) \to \mathscr{D}(\zeta x, \zeta y)$ is well defined by naturality of ζ.

Definition 7.30 is of course motivated by the standard notion of an equivalence between categories. We shall now work towards a model structure that incorporates Dwyer-Kan equivalences as its weak equivalences.

Definition 7.32. For $\mathscr{C}_{\bullet\bullet}$ a Segal space let

$$\mathscr{C}_{\text{equiv}} \hookrightarrow \mathscr{C}_1$$

be the inclusion of the connected components of vertices that are invertible (see Definition 7.29).

The identity morphisms (up to homotopy) of a Segal space are induced by the degeneracy map $\mathscr{C}_{s^0\bullet} \colon \mathscr{C}_{0\bullet} \to \mathscr{C}_{1\bullet}$ and its 0-th component

$$\mathscr{C}_{s^0,0} \colon \mathscr{C}_{0,0} \to \mathscr{C}_{1,0}, \qquad x \mapsto 1_x$$

This therefore turns out to be an inclusion

$$\mathscr{C}_0 \to \mathscr{C}_{\text{equiv}}$$

Definition 7.33. A Segal space $\mathscr{C}$ is called *complete* if the map $\mathscr{C}_0 \to \mathscr{C}_{\text{equiv}}$ is a weak equivalence of simplicial sets.

The idea of the above definition is that if a morphism is an isomorphism then it is already a morphism in $\mathscr{C}_0$. This is somewhat akin to what it means for an ordinary category to be *skeletal*. Recall that a category is called skeletal if all its isomorphisms are identities. However, any (ordinary) category is equivalent to a skeletal one, so this is not really a restriction in general (see the Nlab article on Skeleton).

Example 7.34. Let $\mathscr{C}$ be an (ordinary) category. The simplicial space $\mathfrak{N}\mathscr{C}_{\bullet\star}$ is a complete Segal space if and only if there are no non-identity isomorphisms in $\mathscr{C}$.

Yet again, we want to shift this into a model theoretic picture, encapsulating the notion of complete Segal space by means of a left Bousfield localization. To this end, let us consider the category $\mathfrak{I}(1)$ with two distinct objects and one isomorphism between these. This category is called the *walking isomorphism.* If we map the walking isomorphism into an arbitrary category $\mathscr{C}$, we obtain all the isomorphisms of $\mathscr{C}$. More precisely, we have an isomorphism of categories

$$\mathscr{C}^{\mathfrak{I}[1]} \xrightarrow{\ \cong\ } (\mathscr{C}^{\times})^{[1]}$$

where $\mathscr{C}^{\times}$ is the maximal subgroupoid of $\mathscr{C}$. Rezk, in his paper [31], then proved the following non-trivial theorem:

Theorem 7.35. *A Segal space $\mathscr{C}$ is a complete Segal space if and only if $\mathscr{C}$ is local with respect to the (unique) morphism*

$$\mathfrak{N}(\mathfrak{I}(1))_{\bullet\star} \longrightarrow \Delta^0_{\bullet\star}$$

In other words, $\mathscr{C}$ is complete if and only if

$$\mathbb{R}\mathrm{Map}(\Delta^0_{\bullet\star},\ \mathscr{C}) \longrightarrow \mathbb{R}\mathrm{Map}(\mathfrak{N}(\mathfrak{I}(1))_{\bullet\star},\ \mathscr{C})$$

is a weak equivalence of simplicial sets.

In other words, we can make the following equivalent definition:

Definition 7.36. A *complete Segal space* is a simplicial space $\mathscr{C}\colon \Delta^{\mathrm{op}} \to \mathrm{sSet}$ which satisfies the following conditions:

- *Fibrancy*: $\mathscr{C}$ is fibrant with respect to the model structure on SeSp (in other words, $\mathscr{C}$ is a Segal space).
- *Completeness condition*: $\mathscr{C}$ is local with respect to the unique map ($\Delta_{\bullet\star}$ is terminal)

$$\mathfrak{N}(\mathfrak{I}(1))_{\bullet\star} \to \Delta^0_{\bullet\star}$$

Theorem 7.37. *There is a cartesian model structure on the category of simplicial spaces, denoted by* CSS*, in which all objects are cofibrant and the fibrant objects are precisely the complete Segal spaces. In fact,* CSS *is given by the left Bousfield localization*

$$\mathrm{CSS} = \mathrm{L}_{S'}(\mathrm{Psh}_{\Delta}(\Delta)_{\mathrm{inj}})$$

where S' is the family of morphisms

$$\Delta^a_{\bullet\star} \coprod\nolimits_{\Delta^0_{\bullet\star}} \Delta^b_{\bullet\star} \longrightarrow \Delta^{a+b}_{\bullet\star}$$

along with the unique morphism

$$\mathfrak{N}(\mathfrak{I}[1])_{\bullet\star} \to \Delta^0_{\bullet\star}$$

In particular, the collection of weak equivalences in the model category CSS *is precisely given by the Dwyer-Kan equivalences.*

Finally, we have the following definition:

Definition 7.38. An $(\infty,1)$-category is a fibrant object in CSS, that is, a complete Segal space.

The above theorem also tells us that the model category of complete Segal spaces is a cartesian closed model category. This means that the internal hom functor respects the given mdoel structure, i.e., it is a right Quillen bifunctor. Therefore, we have a notion of a derived hom in the given model structure. Since all objects in CSS are cofibrant, we have

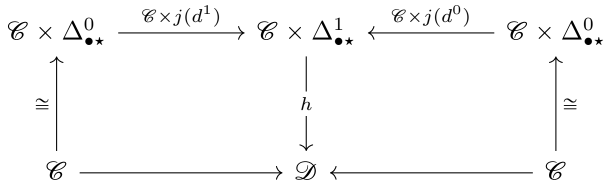
$$\mathbb{R}\mathrm{Hom}(\mathscr{C},\mathscr{D}) \simeq \mathscr{D}^{\mathscr{C}}$$

for all complete Segal spaces $\mathscr{D}$ and all bisimplicial sets $\mathscr{C}$, where $\mathscr{D}^{\mathscr{C}}$ denotes the corresponding internal hom in bisimplicial spaces. In particular, for any pair of ∞-categories $\mathscr{C},\mathscr{D}$ we get an ∞-category of functors $\mathscr{D}^{\mathscr{C}}$:

Definition 7.39. Let $\mathscr{C},\mathscr{D}$ be $(\infty,1)$-categories (complete Segal spaces).

- An ∞-*functor* (or $(\infty,1)$-*functor*) from $\mathscr{C}$ to $\mathscr{D}$ is a natural transformation $\mathscr{C} \to \mathscr{D}$.
- An ∞-*natural transformation* (or $(\infty,1)$-*natural transformation*) between ∞-functors with domain $\mathscr{C}$ and codomain $\mathscr{D}$ is a homotopy $h\colon \mathscr{C} \times \Delta^1_{\bullet\star} \to$

$\mathscr{D}$. The domain and codomain of the ∞-natural transformation h is read off of the commutative diagram

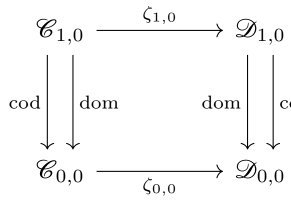


as the bottom left and bottom right maps.

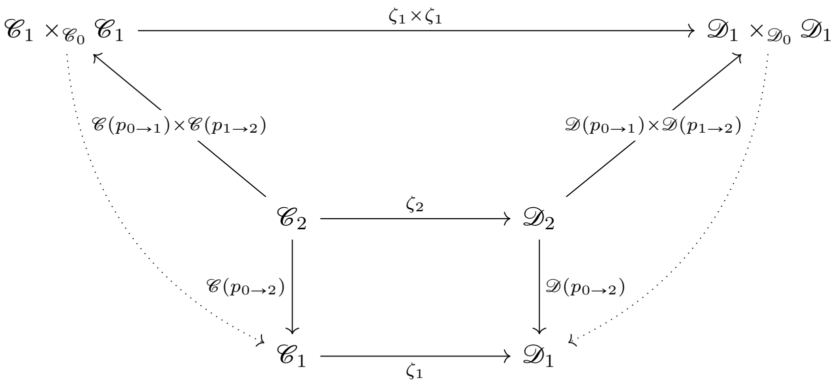
Remark 7.40. Let us study quickly why the definition above really yields the correct notion of ∞-functors: Let $\zeta\colon \mathscr{C} \to \mathscr{D}$ be an ∞-functor between ∞-categories. We then, in particular, have maps on objects and 1-morphisms

$$\zeta_{0,0}\colon \mathscr{C}_{0,0} \to \mathscr{D}_{0,0}, \qquad \zeta_{1,0}\colon \mathscr{C}_{1,0} \to \mathscr{D}_{1,0}$$

We shall abuse notation and always write ζ instead of $\zeta_{0,0}, \zeta_{1,0}$, etc. whenever it is clear from the context. A morphism $(f\colon x \to y) \in \mathscr{C}_{1,0}$ is mapped to a morphism $\zeta f\colon \zeta x \to \zeta y$, which follows from naturality of ζ:



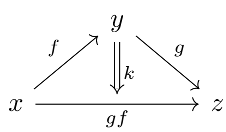
In particular, functoriality follows from commutativity of the diagram



More precisely, let $k \in \mathscr{C}_{2,0}$ be such that

$$(\mathscr{C}p_{0\to 1} \times \mathscr{C}p_{1\to 2})(k) = (f, g) \in \mathscr{C}(x, y, z)$$

or put geometrically



Now ζ induces a 2-simplex $\zeta k \in \mathscr{D}_{2,0}$ which produces the following identities

$$\mathscr{D}_{d^0,0}(\zeta k) = \zeta g, \qquad \mathscr{D}_{d^1,0}(\zeta k) = \zeta(gf), \qquad \mathscr{D}_{d^2,0}(\zeta k) = \zeta f$$

which follows from naturality:

$$\begin{array}{ccc} \mathscr{C}_{2,0} & \xrightarrow{\zeta} & \mathscr{D}_{2,0} \\ \Downarrow\!\Downarrow\!\Downarrow & & \Downarrow\!\Downarrow\!\Downarrow \\ \mathscr{C}_{1,0} & \xrightarrow[\zeta]{} & \mathscr{D}_{1,0} \end{array}$$

Rephrasing this geometrically yields

$$\begin{array}{ccccc} & & \zeta y & & \\ & \nearrow^{\zeta f} & \Downarrow \zeta k & \searrow^{\zeta g} & \\ \zeta x & & \xrightarrow[\zeta(gf)]{} & & \zeta z \end{array}$$

and therefore ζk witnesses the identity $\zeta(fg) = \zeta g \zeta f$. The same procedure works to show that ζ preserves identities.

Remark 7.41. Also the notion of ∞-natural transformations is seen to have the desired outcome. Passing to the homotopy 1-category results in a natural transformation between the induced functors. This is analogous to the construction in Remark 7.13. Indeed, any ∞-natural transformation

$$\begin{array}{ccccc} \mathscr{C} \times \Delta^0 & \xrightarrow{1 \times d^1_{\bullet\star}} & \mathscr{C} \times \Delta^1_{\bullet\star} & \xleftarrow{1 \times d^0_{\bullet\star}} & \mathscr{C} \times \Delta^0 \\ \uparrow \cong & & \downarrow h & & \uparrow \cong \\ \mathscr{C} & \xrightarrow[\zeta_1 := h(1 \times d^1_{\bullet\star})]{} & \mathscr{D} & \xleftarrow[\zeta_2 := h(1 \times d^0_{\bullet\star})]{} & \mathscr{C} \end{array}$$

yields, for every 1-morphism $f \in \mathscr{C}_{1,0}$, a quadruple of maps

$$\Delta^1_{\bullet\star} \times \Delta^0 \underset{1 \times d^0_{\bullet\star}}{\overset{1 \times d^1_{\bullet\star}}{\rightrightarrows}} \Delta^1_{\bullet\star} \times \Delta^1_{\bullet\star} \xrightarrow{f \times 1} \mathscr{C} \times \Delta^1_{\bullet\star} \xrightarrow{h} \mathscr{D}$$

$$\Delta^0 \times \Delta^1_{\bullet\star} \underset{d^0_{\bullet\star} \times 1}{\overset{d^1_{\bullet\star} \times 1}{\rightrightarrows}} \Delta^1_{\bullet\star} \times \Delta^1_{\bullet\star} \xrightarrow{f \times 1} \mathscr{C} \times \Delta^1_{\bullet\star} \xrightarrow{h} \mathscr{D}$$

which in turn gives rise to a square of 1-morphisms in $\mathscr{D}$

$$\begin{array}{ccc} \zeta_1(\mathrm{dom} f) & \xrightarrow{h_{\mathrm{dom} f}} & \zeta_2(\mathrm{dom} f) \\ \downarrow \zeta_1 f & & \downarrow \zeta_2 f \\ \zeta_1(\mathrm{cod} f) & \xrightarrow[h_{\mathrm{cod} f}]{} & \zeta_2(\mathrm{cod} f) \end{array}$$

Again, in order to show that this diagram commutes it is enough to show that $\mathfrak{h}_1(\Delta^1_{\bullet\star} \times \Delta^1_{\bullet\star})$ is the category whose formal graph is a commutative square. This is immediate however, since $\Delta^1_{\bullet\star} \times \Delta^1_{\bullet\star}$ is constant in the second simplicial direction and thus elements in $(\Delta^1_{\bullet\star} \times \Delta^1_{\bullet\star})_{1,1} = \Delta^1_1 \times \Delta^1_1$ really just tell us that both compositions are identical.

From the previous considerations we readily obtain the following:

Corollary 7.42. *Let $\mathscr{C},\mathscr{D}$ be $(\infty,1)$-categories. Then any ∞-functor $\zeta\colon \mathscr{C}\to\mathscr{D}$ induces a functor*

$$\mathfrak{h}_1\zeta\colon \mathfrak{h}_1\mathscr{C}\to\mathfrak{h}_1\mathscr{D}$$

In particular, any ∞-natural transformation $h\colon \zeta_1\to\zeta_2$ induces a natural transformation

$$\mathfrak{h}_1(h)\colon \mathfrak{h}_1\zeta_1\to\mathfrak{h}_1\zeta_2$$

7.3.2. *The Rezk nerve.* We have seen that the ordinary nerve operation applied to some category $\mathscr{C}$ and then viewed as a simplicial space does not in general yield a complete Segal space. However, since complete Segal spaces are our chosen model for $(\infty,1)$-categories we should better provide for a proper inclusion of the category of (small) categories into the the full subcategory of complete Segal spaces. Luckily enough, there is an improved version of the nerve functor called the *Rezk nerve*: Let $(\mathscr{C},\mathscr{W})$ be a pair consisting of a category $\mathscr{C}$ together with a wide subcategory $\mathscr{W}$ of weak equivalences. The simplicial space $\mathfrak{N}^\infty(\mathscr{C},\mathscr{W})$, called *Rezk nerve* or *classification diagram* of $(\mathscr{C},\mathscr{W})$, which, if evaluated in the first simplicial direction by $[m]\in\Delta$ yields as a simplicial set the (usual) nerve of the wide subcategory $\mathrm{we}(\mathscr{C}^{[m]})\subset\mathscr{C}^{[m]}$ which has as its morphisms only those natural transformations which are weak equivalences objectwise. In other words,

$$(\mathfrak{N}^\infty(\mathscr{C},\mathscr{W}))_m := \mathfrak{N}(\mathrm{we}(\mathscr{C}^{[m]}))$$

Example 7.45. For a category $\mathscr{C}$ we may apply the Rezk nerve to the pair $(\mathscr{C},\mathscr{C}^\times)$, where $\mathscr{C}^\times$ is the maximal subgroupoid of $\mathscr{C}$. This yields a functor

$$\mathfrak{N}^\infty\colon \mathrm{Cat}\to\mathrm{Psh}_\Delta(\Delta),\qquad \mathscr{C}\mapsto\mathfrak{N}(\mathrm{iso}(\mathscr{C}^\bullet))_\bullet$$

The preceding example is very important:

Theorem 7.46 ([31] Theorem 3.7). *The Rezk nerve*

$$\mathfrak{N}^\infty\colon \mathrm{Cat}\to\mathrm{Psh}_\Delta(\Delta)$$

is fully faithful. Moreover, there are natural isomorphisms of bisimplicial sets

$$\mathfrak{N}^\infty(\mathscr{C}\times\mathscr{D})\cong\mathfrak{N}^\infty\mathscr{C}\times\mathfrak{N}^\infty\mathscr{D},\qquad \mathfrak{N}^\infty(\mathscr{D}^{\mathscr{C}})\cong(\mathfrak{N}^\infty\mathscr{D})^{\mathfrak{N}^\infty\mathscr{C}}$$

for categories $\mathscr{C},\mathscr{D}$. In particular, a functor $f\colon\mathscr{C}\to\mathscr{D}$ is an equivalence of categories if and only if $\mathfrak{N}^\infty f$ is a weak equivalence of bisimplicial sets (wrt. the injective model structure).

Remark 7.47. It may be shown that $\mathfrak{N}^\infty(\mathscr{C},\mathscr{W})$ satisfies Segal's special Δ-conditions (this follows from the 2-out-of 3 property, see [8]). Moreover, $\mathfrak{N}^\infty(\mathscr{C},\mathscr{W})$ is complete if and only if the homotopical category $\mathscr{C}$ is *saturated*, that is, a morphism in $\mathscr{C}$ is a weak equivalence if and only if it is an isomorphism in the homotopy category. However, $\mathfrak{N}^\infty(\mathscr{C},\mathscr{W})$ does not necessarily satisfy the fibrancy condition to make it into a complete Segal space. Nonetheless, treating $\mathfrak{N}^\infty(\mathscr{C},\mathscr{W})$ as an ∞-category, we obtain the following interpretation:

- Objects are precisely the elements in
$$\mathfrak{N}^\infty(\mathscr{C},\mathscr{W})_{0,0}=\mathfrak{N}(\mathscr{W})_0=\mathrm{Ob}\mathscr{C}$$
- 1-morphisms are elements in
$$\mathfrak{N}^\infty(\mathscr{C},\mathscr{W})_{1,0}=\mathfrak{N}(\mathrm{we}(\mathscr{C}^{[1]}))_0=\mathrm{Mor}\mathscr{C}$$
which is what we would expect of any reasonable way to embed an arbitrary category into the setting of ∞-categories.

If we really want to realize $\mathfrak{N}^\infty(\mathscr{C},\mathscr{W})$ as an $(\infty,1)$-category, we have to force the fibrancy condition by taking some fibrant replacement of $\mathfrak{N}^\infty(\mathscr{C},\mathscr{W})$. The resulting complete Segal space will again be denoted by $\mathfrak{N}^\infty(\mathscr{C},\mathscr{W})$.

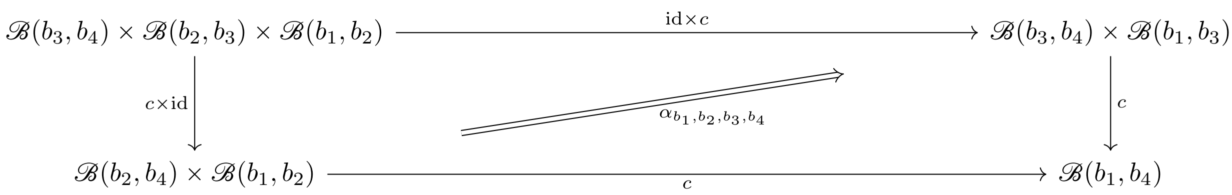
7.4. $(\infty, 2)$**-Categories.** Before getting into the meat of this section, let us start off by recalling the notion of *bicategory*: Let us briefly recall the definition of a *bicategory*:

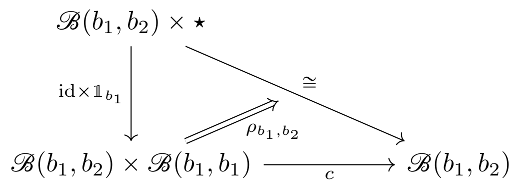
Definition 7.48. A *bicategory* is a tuple

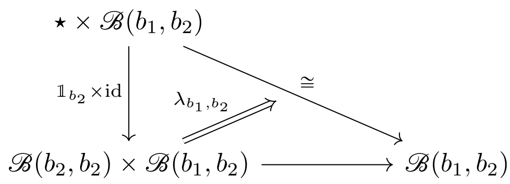
$$(\mathscr{B}, 1, c, \alpha, \lambda, \rho)$$

comprised of the following data:

- $\mathscr{B}$ comes equipped with a collection of objects $\mathrm{Ob}\mathscr{B} = \mathscr{B}_0$.
- For each pair of objects $b, b' \in \mathscr{B}$ we have a hom-category $\mathscr{B}(x, y)$. The identity morphism in $\mathscr{B}(x, y)$ for an object $f \in \mathscr{B}(x, y)$ will be denoted by 1_f.
- The objects in $\mathscr{B}(x, y)$ are called 1-morphisms, and the collection of all such 1-morphisms is denoted by $\mathscr{B}_1$. The morphisms in the category $\mathscr{B}(x, y)$ are called 2-morphisms.
- We have composition functors
$$\begin{aligned} c\colon \mathscr{B}(b_2, b_3) \times \mathscr{B}(b_1, b_2) &\to \mathscr{B}(b_1, b_3) \\ (g, f) &\mapsto g \square f \\ (\beta, \alpha) &\mapsto \beta\alpha \end{aligned}$$
for all objects $b_1, b_2, b_3, b_4 \in \mathscr{B}_0$ and unit functors
$$\mathbb{1}_b\colon \star \to \mathscr{B}(b, b)$$
from the terminal category $\star$ into the category $\mathscr{B}(b, b)$, which pick out an identity 1-morphism $\mathbb{1}_b$ for all $b \in \mathscr{B}_0$.
- We have natural isomorphisms



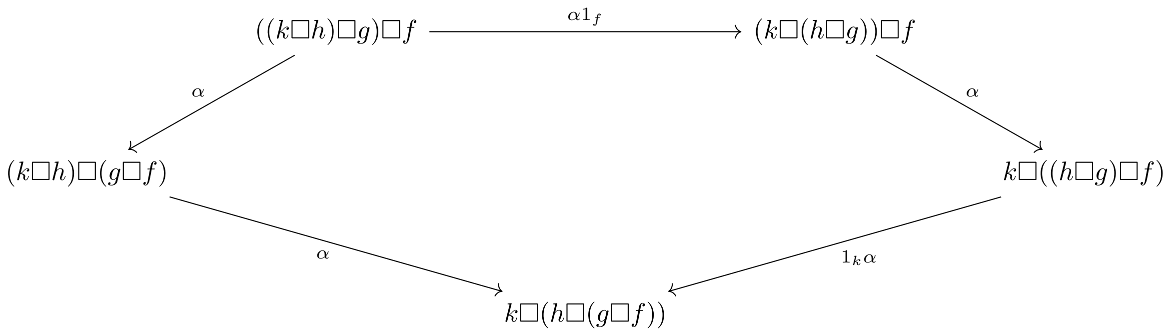


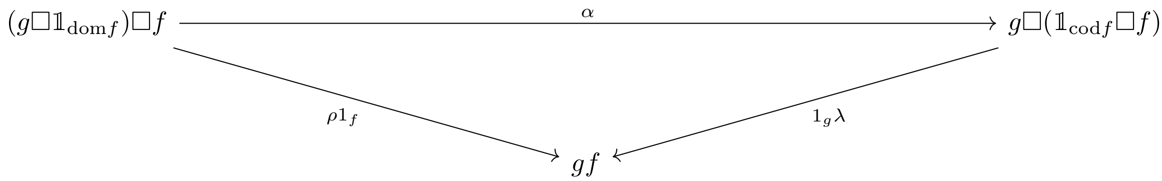


In particular, this gives us invertible 2-morphisms

$$\begin{aligned} \alpha_{hgf}\colon (h \square g) \square f &\xrightarrow{\cong} h \square (g \square f) \\ \rho_f\colon f \square \mathbb{1}_{\mathrm{dom} f} &\xrightarrow{\cong} f \\ \lambda_f\colon \mathbb{1}_{\mathrm{cod} f} \square f &\xrightarrow{\cong} f \end{aligned}$$

- In particular, we demand that the following diagrams commute:





Remark 7.49. Any monoidal category $\mathscr{C}$ may be interpreted as a bicategory $B\mathscr{C}$ as follows:

- $B\mathscr{C}$ has only one object $\star$.
- The category of 1-morphisms $B\mathscr{C}(\star,\star)$ is defined to be $\mathscr{C}$.
- The composition law is given by the tensor product:

$$\otimes\colon \mathscr{C}\times\mathscr{C}\to\mathscr{C}$$

Conversely, any bicategory with only a single object is canonically a monoidal category.

An $(\infty,2)$-category will be a homotopical analogue of a bicategory with infinitely many layers of morphisms, yet only two of these layers are non-trivial. To this end, we continue in the exact same manner as in the previous chapter. First we shall add another categorical layer by adding one more simplicial level:

Definition 7.50. The category of *bisimplicial spaces* is the category of presheaves $\mathrm{Psh}_\Delta(\Delta^{\times 2})$.

Notation 7.51. We shall again write $\mathrm{Hom}(-,-)$ for the hom-set bifunctor of the category of bisimplicial spaces.

We may then consider the projections $\pi_i\colon \Delta^{\times 3}\to\Delta$ for $i=1,2,3$. These give rise to maps

$$\mathrm{sSet}\xrightarrow{\ \pi_i^\star\ }\mathrm{Psh}_\Delta(\Delta^{\times 2})$$

Extending on the notation we introduced in the previous chapter, a bisimplicial space $X\in\mathrm{Psh}_\Delta(\Delta^{\times 2})$ may be written as $X=X_{\bullet\bullet\bullet}$. For the simplicial set Δ^n we then have three possibilities of viewing it as a bisimplicial space: $\Delta^n_{\bullet\star\star}$, $\Delta^n_{\star\bullet\star}$ and $\Delta^n_{\star\star\bullet}$. The category of bisimplicial spaces is enriched over sSet by defining

$$\mathrm{Map}(X,Y) := \mathrm{Hom}(X\times\pi_3^\star\,\text{よ}_\Delta, Y)$$

for all $X,Y\in\mathrm{Psh}_\Delta(\Delta^{\times 2})$. We may then apply $\pi_1^\star$ and $\pi_2^\star$ to the induced maps

$$\Delta^a\coprod_{\Delta^0}\Delta^b \longrightarrow \Delta^{a+b}$$

so as to obtain morphisms

$$\Delta^a_{\bullet\star\star} \coprod_{\Delta^0_{\bullet\star\star}} \Delta^b_{\bullet\star\star} \xrightarrow{\varphi^{a,b}_{\bullet\star\star}} \Delta^{a+b}_{\bullet\star\star}$$

$$\Delta^a_{\star\bullet\star} \coprod_{\Delta^0_{\star\bullet\star}} \Delta^b_{\star\bullet\star} \xrightarrow{\varphi^{a,b}_{\star\bullet\star}} \Delta^{a+b}_{\star\bullet\star}$$

In particular, both $\Delta^0_{\bullet\star\bullet}$ and $\Delta^0_{\star\bullet\star}$ are terminal, so there are unique morphisms

$$\mathfrak{N}(\mathfrak{I}[1])_{\bullet\star\star} \xrightarrow{c_{\bullet\star\star}} \Delta^0_{\bullet\star\star}$$

$$\mathfrak{N}(\mathfrak{I}[1])_{\star\bullet\star} \xrightarrow{c_{\star\bullet\star}} \Delta^0_{\star\bullet\star}$$

Definition 7.52. A *complete double Segal space* is a bisimplicial space $\mathscr{C}\colon (\Delta^{\times 2})^{\mathrm{op}} \to$ sSet such that

- *Fibrancy*: $\mathscr{C}$ is fibrant with respect to the injective model structure on $\mathrm{Psh}_\Delta(\Delta^{\times 2})$.
- *Segal's special* Δ *and completeness condition*: $\mathscr{C}$ is local with respect to the maps

$$(\Delta^a_{\bullet\star\star} \coprod_{\Delta^0_{\bullet\star\star}} \Delta^b_{\bullet\star\star}) \times \Delta^c_{\star\bullet\star} \xrightarrow{\varphi^{a,b}_{\bullet\star\star} \times \Delta^c_{\star\bullet\star}} \Delta^{a+b}_{\bullet\star\star} \times \Delta^c_{\star\bullet\star}$$

$$(\Delta^a_{\star\bullet\star} \coprod_{\Delta^0_{\star\bullet\star}} \Delta^b_{\star\bullet\star}) \times \Delta^c_{\bullet\star\star} \xrightarrow{\varphi^{a,b}_{\star\bullet\star} \times \Delta^c_{\bullet\star\star}} \Delta^{a+b}_{\star\bullet\star} \times \Delta^c_{\bullet\star\star}$$

$$\mathfrak{N}(\mathfrak{I}[1])_{\bullet\star\star} \xrightarrow{c_{\bullet\star\star}} \Delta^0_{\bullet\star\star}$$

$$\Delta^c_{\bullet\star\star} \times \mathfrak{N}(\mathfrak{I}[1])_{\star\bullet\star} \xrightarrow{\Delta^c_{\bullet\star\star} \times c_{\star\bullet\star}} \Delta^c_{\bullet\star\star} \times \Delta^0_{\star\bullet\star}$$

  for all $a, b, c \in \mathbb{N}$.

Remark 7.53. One may certainly drop the completeness conditions to arrive at a notion of *double Segal space*.

Remark 7.54. Let us break down the main ideas of Definition 7.52: Segal's special Δ condition boils down to the statement that both $\mathscr{C}_{c\bullet\bullet}$ and $\mathscr{C}_{\bullet c\bullet}$ are Segal spaces, i.e., both maps

$$\mathscr{C}_{c,a+b} \xrightarrow{\simeq} \mathscr{C}_{c,a} \times_{\mathscr{C}_{c,0}} \mathscr{C}_{c,b}$$

$$\mathscr{C}_{a+b,c} \xrightarrow{\simeq} \mathscr{C}_{a,c} \times_{\mathscr{C}_{0,c}} \mathscr{C}_{b,c}$$

are trivial Kan fibrations for all $a, b, c \in \mathbb{N}$. The completeness condition boils down to saying that both $\mathscr{C}_{\bullet 0\bullet}$ and $\mathscr{C}_{c\bullet\bullet}$ are complete Segal spaces for all $c \in \mathbb{N}$.

Theorem 7.55. *There is a model structure* $\mathrm{CSS}_2^{\mathrm{uple}}$ *on the category of bisimplicial spaces in which the fibrant objects are precisely the complete double Segal spaces. In fact, this model structure is obtained by means of the left Bousfield localization*

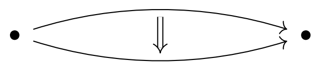
$$\mathrm{CSS}_2^{\mathrm{uple}} := \mathrm{L}_S(\mathrm{Psh}_\Delta(\Delta^{\times 2})_{\mathrm{inj}})$$

where S *is the family of morphisms as given in Definition* 7.52.

Yet again we find ourselves not fully satisfied with (complete) double Segal spaces. The problem here is similar as the one we had with Segal spaces. A double Segal space encodes the information of a homotopical double category. Recall that a double category has objects, horizontal morphisms, vertical morphisms and 2-morphisms (squares). Suppose $\mathscr{C}$ is a (complete) double Segal space. We think of $\mathscr{C}_{0,0}$ as the *space of objects*, $\mathscr{C}_{0,1}$ as the *space of vertical morphisms*, $\mathscr{C}_{1,0}$ as the *space of horizontal morphisms*, and $\mathscr{C}_{1,1}$ as the space of squares. Indeed, the 2-morphisms (squares) are encoded by diagrams of the form:

$$\begin{array}{ccc}
\mathscr{C}_{0,0} \ni \mathscr{C}_{d^1,d^1}(\psi) & \xrightarrow{\mathscr{C}_{1,d^1}(\psi)\in\mathscr{C}_{1,0}} & \mathscr{C}_{d^0,d^1}(\psi) \in \mathscr{C}_{0,0} \\
{\scriptstyle \mathscr{C}_{d^1,1}(\psi)\in\mathscr{C}_{0,1}}\big\downarrow & \Downarrow{\scriptstyle \psi\in\mathscr{C}_{1,1}} & \big\downarrow{\scriptstyle \mathscr{C}_{d^0,1}(\psi)\in\mathscr{C}_{0,1}} \\
\mathscr{C}_{0,0} \ni \mathscr{C}_{d^1,d^0}(\psi) & \xrightarrow[\mathscr{C}_{1,d^0}(\psi)\in\mathscr{C}_{1,0}]{} & \mathscr{C}_{d^0,d^0}(\psi) \in \mathscr{C}_{0,0}
\end{array}$$

On the other hand, when we picture a 2-morphism in a 2-category we liken it more to something as



In other words, we do not want to have non-trivial vertical morphisms, but only horizontal ones. Therefore, it is natural to demand or try to force the simplicial space $\mathscr{C}_{0\bullet\bullet}$ to be *essentially constant.* Yet again, we shall encode this by means of a left Bousfield localization. For $\mathbf{m} = ([m_1],[m_2]) \in \Delta^{\times 2}$, let $\hat{\mathbf{m}} \in \Delta^{\times 2}$ be the bisimplex defined by

$$[\hat{m}_i] = \begin{cases} [0], & \text{if } \exists j < i\colon \text{ with } m_j = 0 \\ [m_i], & \text{else} \end{cases}$$

for $i = 1, 2$. More concretely, if $m_1 = 0$, then $\hat{\mathbf{m}} = \mathbf{0}$, but if $m_1 \neq 0$, then $\mathbf{m} = \hat{\mathbf{m}}$. This gives rise to a canonical map

$$\mathbf{m} \longrightarrow \hat{\mathbf{m}}$$

which maps $m_i \mapsto \hat{m}_i$ for all i. In turn, we may plug this map into the functor $\pi^\star_{1,2}\,\text{よ}_{\Delta^{\times 2}}$, where $\pi_{1,2}\colon \Delta^{\times 3} \to \Delta$ is the projection onto the first two factors, to give us a morphism

$$\pi^\star_{1,2}\,\text{よ}_{\Delta^{\times 2}}\mathbf{m} \longrightarrow \pi^\star_{1,2}\,\text{よ}_{\Delta^{\times 2}}\hat{\mathbf{m}}$$

Definition 7.56. A *2-fold complete Segal space* is a bisimplicial space $\mathscr{C}\colon (\Delta^{\times 2})^{\mathrm{op}} \to$ sSet such that

- $\mathscr{C}$ is fibrant with respect to the model structure as given in Theorem 7.55.
- $\mathscr{C}$ is local with respect to the family of morphisms

$$\pi^\star_{1,2}\,\text{よ}_{\Delta^{\times 2}}\mathbf{m} \longrightarrow \pi^\star_{1,2}\,\text{よ}_{\Delta^{\times 2}}\hat{\mathbf{m}}$$

  for all $\mathbf{m} \in \Delta^{\times 2}$.

Remark 7.57. Dropping yet again the completeness condition in the above definition yields the notion of a d-fold Segal space.

Remark 7.58. $\mathscr{C}$ being local with respect to the above family of maps means that the induced maps

$$\mathbb{R}\mathrm{Map}(\pi^\star_{1,2}\,\text{よ}_{\Delta^{\times 2}}\hat{\mathbf{m}}) \longrightarrow \mathbb{R}\mathrm{Map}(\pi^\star_{1,2}\,\text{よ}_{\Delta^{\times 2}}\mathbf{m}, \mathscr{C})$$

are weak equivalences of simplicial sets. However, this exactly boils down to saying that we have weak equivalences $\mathscr{C}_{0,m,\bullet} \xrightarrow{\simeq} \mathscr{C}_{0,0,\bullet}$ for all $m \in \mathbb{N}$. This is what we meant by saying that $\mathscr{C}_{0,\bullet,\bullet}$ is essentially constant.

Theorem 7.59. *There is a model structure* $\mathrm{CSS}_2^{\mathrm{glob}}$ *on the category of bisimplicial spaces in which the fibrant objects are precisely the 2-fold complete Segal spaces. In fact, this model structure is obtained by means of the left Bousfield localization*

$$\mathrm{CSS}_2^{\mathrm{glob}} := \mathrm{L}_S(\mathrm{CSS}_2^{\mathrm{uple}})$$

where S is the family of morphisms as given in Definition 7.56.

Remark 7.60. The model structure on $\mathrm{CSS}_2^{\mathrm{glob}}$ is referred to as the *globular model structure*, while the model structure on $\mathrm{CSS}_2^{\mathrm{uple}}$ is called the *multiple model structure.*

Definition 7.61. An $(\infty, 2)$-category is a fibrant object in $\mathrm{CSS}_2^{\mathrm{glob}}$.

In a 2-fold Segal space we have

$$\begin{array}{ccc}
\mathscr{C}_{0,0} \ni \mathscr{C}_{d^1,d^1}(\psi) & \xrightarrow{\mathscr{C}_{1,d^1}(\psi)\in\mathscr{C}_{1,0}} & \mathscr{C}_{d^0,d^1}(\psi) \in \mathscr{C}_{0,0} \\
{\scriptstyle \mathscr{C}_{d^1,1}(\psi)\in\mathscr{C}_{0,1}\simeq\mathscr{C}_{0,0}} \Big\downarrow & \Downarrow {\scriptstyle \psi\in\mathscr{C}_{1,1}} & \Big\downarrow {\scriptstyle \mathscr{C}_{d^0,1}(\psi)\in\mathscr{C}_{0,1}\simeq\mathscr{C}_{0,0}} \\
\mathscr{C}_{0,0} \ni \mathscr{C}_{d^1,d^0}(\psi) & \xrightarrow[\mathscr{C}_{1,d^0}(\psi)\in\mathscr{C}_{1,0}]{} & \mathscr{C}_{d^0,d^0}(\psi) \in \mathscr{C}_{0,0}
\end{array}$$

By the conditions imposed on what it means to be a 2-fold Segal space, the dotted vertical arrows are essentially forced to be just identity maps, up to homotopy, since $\mathscr{C}_{0,1,\bullet} \simeq \mathscr{C}_{0,0,\bullet}$.

Definition 7.62. The *homotopy bicategory* $\mathfrak{h}_2\mathscr{C}$ of a 2-fold (complete) Segal space $\mathscr{C} = \mathscr{C}_{\bullet\bullet\bullet}$ is the bicategory which has as objects the set $\mathscr{C}_{0,0,0}$ and for $x, y \in \mathscr{C}_{0,0,0}$ the hom-category

$$\mathfrak{h}_2\mathscr{C}(x,y) := \mathfrak{h}_1(\mathscr{C}(x,y))$$

where $\mathscr{C}(x,y)_{\bullet\bullet}$ is the complete Segal space defined by

$$\Delta^0 \times^x_{\mathscr{C}_{0,\bullet,\bullet}} \mathscr{C}_{1,\bullet,\bullet} \times^y_{\mathscr{C}_{0,\bullet,\bullet}} \Delta^0$$

where Δ^0 denotes the terminal object in $\mathrm{Psh}_\Delta(\Delta^{\times 2})$. Horizontal composition is then defined by means of the following dotted arrow

$$\begin{array}{ccc}
 & & \left(\Delta^0 \times^x_{\mathscr{C}_{0,\bullet,\bullet}} \mathscr{C}_{1,\bullet,\bullet} \times^y_{\mathscr{C}_{0,\bullet,\bullet}} \Delta^0\right) \times \left(\Delta^0 \times^y_{\mathscr{C}_{0,\bullet,\bullet}} \mathscr{C}_{1,\bullet,\bullet} \times^z_{\mathscr{C}_{0,\bullet,\bullet}} \Delta^0\right) \\
 & \swarrow & \Big\downarrow \\
\left(\Delta^0 \times^x_{\mathscr{C}_{0,\bullet,\bullet}} \mathscr{C}_{1,\bullet,\bullet} \times^z_{\mathscr{C}_{0,\bullet,\bullet}} \Delta^0\right) & & \left(\Delta^0 \times^x_{\mathscr{C}_{0,\bullet,\bullet}} \mathscr{C}_{1,\bullet,\bullet} \times_{\mathscr{C}_{0,\bullet,\bullet}} \mathscr{C}_{1,\bullet,\bullet} \times^z_{\mathscr{C}_{0,\bullet,\bullet}} \Delta^0\right) \\
 & \nwarrow & \Big\uparrow \simeq \\
 & & \Delta^0 \times^x_{\mathscr{C}_{0,\bullet,\bullet}} \mathscr{C}_{2,\bullet,\bullet} \times^z_{\mathscr{C}_{0,\bullet,\bullet}} \Delta^0
\end{array}$$

Remark 7.63. According to [8] the previous definition indeed yields a bicategory.

7.5. (∞, d)**-Categories.** At this point the idea for (∞, d)-categories should be clear. We add as many new simplicial layers as needed so as to capture all the necessary complexity of an (∞, d)-category.

Definition 7.64. A *d-dimensional simplicial space* is a functor $X\colon (\Delta^{\times d})^{\mathrm{op}} \to \mathrm{sSet}$. The *category of d-dimensional simplicial spaces* is the category of simplicial presheaves $\mathrm{Psh}_\Delta(\Delta^{\times d})$.

Notation 7.65. It is clear now that notation of the form

$$\Delta^n_{\bullet\star\dots\star}, \qquad \Delta^n_{\star\bullet\dots\star}, \qquad \dots$$

is not so lucid anymore. Hence we shall introduce the following: Let $A \subset \{1, \dots, d\}$ and consider the functor category $\Delta^A := \mathrm{Fun}(A, \Delta)$, where A is viewed as a discrete category. There is an isomorphism of categories

$$\Delta^A \xrightarrow{\cong} \Delta^{\times |A|}$$

which takes a functor $\mathbf{n}\colon A \to \Delta$ to the multisimplex $(\mathbf{n}(a))_{a\in A}$. Essentially, Δ^A is the corresponding sub-product of $\Delta^{\times d}$, where the product is only taken over the elements of A. We may then consider the projection $\pi_{\Delta^A}\colon \Delta^{\times d} \to \Delta^A$ and define the map j_{Δ^A} as the composition:

$$\Delta^A \xhookrightarrow{\text{よ}} \mathrm{Psh}(\Delta^A) \xrightarrow{\pi^\star_{\Delta^A}} \mathrm{Psh}(\Delta^{\times d}) \xrightarrow{\tau^\star_\Delta} \mathrm{Psh}_\Delta(\Delta^{\times d})$$

where $\tau_\Delta\colon \Delta^{\times d} \times \Delta \to \Delta$ denotes the projection onto the last factor.

In particular, if $\pi_{\Delta,k}\colon \Delta^{\times d} \to \Delta$ is the projection onto the k-th factor (i.e. $\pi_{\Delta,k} = \pi_{\Delta^{\{k\}}}$), then define the composition

$$\Delta \xhookrightarrow{\text{よ}} \mathrm{sSet} \xrightarrow{\pi^\star_{\Delta,k}} \mathrm{Psh}(\Delta^{\times d}) \xrightarrow{\tau^\star_\Delta} \mathrm{Psh}_\Delta(\Delta^{\times d})$$

to be $j_{\Delta,k}$. Also, for future reference, if $A \subset \{1, \dots, d\}$, then define $A^{\mathfrak{c}} := \{1, \dots, d\}\backslash A$ to be the complement of A in $\{1, \dots, d\}$. With this notation in hand, we may for example define partial evaluation of $X \in \mathrm{Psh}_\Delta(\Delta^{\times d})$ at an object $\mathbf{n} \in \Delta^A$ denoted $X(\mathbf{n}) \in \mathrm{Psh}_\Delta(\Delta^{A^{\mathfrak{c}}})$. Moreover, we may sometimes just write j for the above embedding without referring to the explicit subfactor, if there is no danger of ambiguity.

Yet again the commutative square

$$\begin{array}{ccc} [a+b] & \xleftarrow{p_{a\to\dots\to a+b}} & [b] \\ {\scriptstyle p_{0\to\dots\to a}}\uparrow & & \uparrow{\scriptstyle p_0} \\ [a] & \xleftarrow[p_a]{} & [0] \end{array}$$

induces maps

$$j_{\Delta,k}[a] \coprod_{j_{\Delta,k}[0]} j_{\Delta,k}[b] \xrightarrow{\varphi^{a,b}} j_{\Delta,k}[a+b]$$

Moreover, for each $1 \leqslant k \leqslant d$ there is a unique map

$$E_k \xrightarrow{c_k} j_{\Delta,k}[0]$$

where

$$E_k := j_{\Delta,k}\mathfrak{N}(\mathfrak{I}[1])$$

Finally, for a multisimplex $\mathbf{m} \in \Delta^{\times d}$ define a multisimplex $\widehat{\mathbf{m}} \in \Delta^{\times d}$ with components

$$[\widehat{m}_i] = \begin{cases} [0], & \text{if } \exists j < i\colon \text{ with } m_j = 0 \\ [m_i], & \text{else} \end{cases}$$

This yields canonical maps

$$\mathbf{m} \longrightarrow \widehat{\mathbf{m}}$$

and applying the functor $j\colon \Delta^{\times d} \to \mathrm{Psh}_\Delta(\Delta^{\times d})$, we get morphisms

$$j\mathbf{m} \longrightarrow j\widehat{\mathbf{m}}$$

Definition 7.66. A *d-fold complete Segal space* is a fibrant object $\mathscr{C} \in \mathrm{Psh}_\Delta(\Delta^{\times d})_{\mathrm{inj}}$ such that

- *Segal's special* Δ*-condition*: $\mathscr{C}$ is local with respect to the family of maps

$$j\mathbf{n} \times \left(j_{\Delta,k}[a] \coprod\nolimits_{j_{\Delta,k}[0]} j_{\Delta,k}[b]\right) \xrightarrow{\quad j\mathbf{n}\times\varphi^{a,b,k}\quad} j\mathbf{n} \times j_{\Delta,k}[a+b]$$

  for all $a, b \in \mathbb{N}, \mathbf{n} \in \Delta^{\{k\}^c}$ and for all $1 \leqslant k \leqslant d$, where we recall that $\{k\}^c := \{1, \ldots, d\} \backslash \{k\}$.
- *Completeness condition*: $\mathscr{C}$ is local with respect to the family of maps

$$j\mathbf{n} \times E_k \xrightarrow{\quad j\mathbf{n}\times c_k\quad} j\mathbf{n} \times j_{\Delta,k}[0]$$

  for all $1 \leqslant k \leqslant d$ and $\mathbf{n} \in \Delta^{A_{k-1}}$, where $A_{k-1} := \{1, \ldots, k-1\}$.
- *Globularity*: X is local with respect to the family of maps

$$j\mathbf{m} \longrightarrow j\widehat{\mathbf{m}}$$

  for all multisimplices $\mathbf{m} \in \Delta^{\times d}$.

Remark 7.67. There are obvious definitions for what d-fold Segal spaces, d-uple Segal spaces, d-uple complete Segal spaces are.

Remark 7.68. Recall that any category of presheaves is cartesian closed with the induced cartesian structure from Set. This implies that the product $c \times -$ is a left adjoint functor (for any object c in such a presheaf category) and therefore preserves colimits. In particular, we have

$$j\mathbf{n} \times \left(j_{\Delta,k}[a] \coprod_{j_{\Delta,k}[0]} j_{\Delta,k}[b]\right) \cong j(\mathbf{n}, a) \coprod_{j(\mathbf{n},0)} j(\mathbf{n}, b)$$

for all $\mathbf{n} \in \Delta^{\{k\}^c}$ and all $[a], [b] \in \Delta$, where $(\mathbf{n}, a)$ is the multisimplex $\{1, \ldots, d\} \to \Delta$ such that $(\mathbf{n}, a)|_{\{k\}^c} = \mathbf{n}$ and $(\mathbf{n}, a)|_{\{k\}} = [a]$ and analogously for $(\mathbf{n}, b)$. In particular, this implies that Segal's special Δ-condition boils down to there being weak equivalences

$$\mathscr{C}(\mathbf{n}, a+b) \xrightarrow{\quad\simeq\quad} \mathscr{C}(\mathbf{n}, a) \times_{\mathscr{C}(\mathbf{n},0)} \mathscr{C}(\mathbf{n}, b)$$

The completeness condition, on the other hand, boils down to saying that each bisimplicial space $\mathscr{C}(\mathbf{n}, -, \mathbf{0}, -)$, with $\mathbf{0} \in \Delta^{\times(d-k)}$ the zero simplex and $\mathbf{n} \in \Delta^{A_k}$, is a complete Segal space. The globularity condition forces that if the k-th component of the multisimplex $\mathbf{m} \in \Delta^{\times d}$ is 0, then we have a weak equivalence

$$\mathscr{C}_{m_1,\ldots,m_{k-1},0,\ldots,0,\bullet} \xrightarrow{\quad\simeq\quad} \mathscr{C}_{\mathbf{m},\bullet}$$

Theorem 7.69. *There are model structures* $\mathrm{CSS}_d^{\mathrm{glob}}$ *and* $\mathrm{CSS}_d^{\mathrm{uple}}$ *on the category of* d*-dimensional simplicial spaces in which the fibrant objects are precisely the* d*-fold complete Segal spaces and the* d*-uple complete Segal spaces, respectively. Both these model structures are defined by means of the corresponding left Bousfield localizations.*

Definition 7.70. An (∞, d)-category is a fibrant object in $\mathrm{CSS}_d^{\mathrm{glob}}$ (or a fibrant object in $\mathrm{CSS}_d^{\mathrm{uple}}$ if we distinguish between *globular* infinity categories and *multiple* infinity categories).

Notation 7.71. We shall write

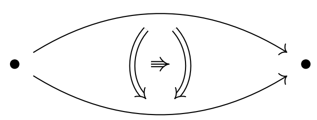
$$\mathrm{Cat}_{(\infty,d)}^{\mathrm{glob}} := \mathrm{CSS}_d^{\mathrm{glob}}$$
$$\mathrm{Cat}_{(\infty,d)}^{\mathrm{uple}} := \mathrm{CSS}_d^{\mathrm{uple}}$$

In particular, for $d = 0$ we have

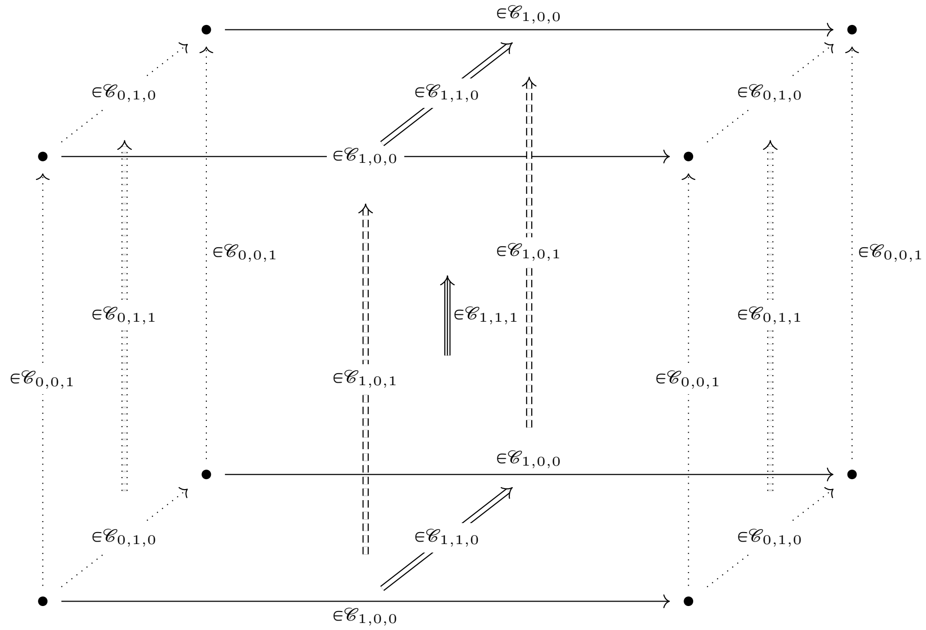
$$\mathrm{Cat}_{(\infty,0)} := \mathrm{Grpd}_\infty := \mathrm{sSet}_{\mathrm{Quillen}}$$

7.5.1. *Interpretation of* d*-fold Segal spaces as higher categories.* How exactly does a d-fold complete Segal space really encode what it should mean to be an (∞, d)-category? The first condition in Definition 7.66 means that there are d different directions in which we can compose. An element of $\mathscr{C}_{\mathbf{k},0}$, with $\mathbf{k} \in \Delta^{\times d}$, should be thought of as composition consisting of k_i morphisms in the i-th direction. The third condition (*globularity condition*) ensures that any d-morphism has as source and target two $(d-1)$-morphisms which themselves have the same source and target (up to homotopy). In general, if we have a d-fold (or d-uple) Segal space $\mathscr{C}$, we should think of the set of 0-simplices of the simplicial set $\mathscr{C}_{\mathbf{0}}$, with $\mathbf{0} \in \Delta^{\times d}$ the zero-multisimplex, as the objects of our category, and vertices of the simplicial set $\mathscr{C}_{\mathbf{1}_i,\mathbf{0}_{d-i}}$ as i-morphisms for $1 \leqslant i \leqslant d$, where $\mathbf{1}_i \in \Delta^{\{1,\dots,i\}}$ with $\mathbf{1}_i(j) = 1$ for all j, and $\mathbf{0}_{d-i} \in \Delta^{\times(d-i)}$ the corresponding zero-multisimplex. In the case of an uple (∞, d)-category, we have several different kinds of i-morphisms. Indeed, for each subset $A \subset \{1, \dots, d\}$ with $|A| = i$, the vertices of the simplicial set $\mathscr{C}_{\mathbf{1}_A,\mathbf{0}_{d-|A|}}$ also form i-morphisms, where $\mathbf{1}_A \in \Delta^A$ with $\mathbf{1}_A(a) = 1$ for all a. In the globular case, however, the vertices of $\mathscr{C}_{\mathbf{1}_i,\mathbf{0}_{d-i}}$ are the only i-morphisms. In both cases, the vertices of the Kan complex $\mathscr{C}_{\mathbf{1}_d}$ yield the collection of d-morphisms in $\mathscr{C}$ and then higher morphisms are given by the morphisms of this Kan complex. In particular, $(d+1)$-morphisms in $\mathscr{C}$ are given by elements of the set $\mathscr{C}_{\mathbf{1}_d,1}$, while $(d+2)$-morphisms are given by elements $\mathscr{C}_{\mathbf{1}_d,2}$ and so on.

Example 7.72. A 3-morphism in a tricategory may be depicted as



whereas a 3-morphism in a 3-fold Segal space $\mathscr{C}$ (an element of the set $\mathscr{C}_{1,1,1,0}$) may be depicted by



Here the dotted arrows are vertices in $\mathscr{C}_{0,1,1} \simeq \mathscr{C}_{0,0,1} \simeq \mathscr{C}_{0,1,0} \simeq \mathscr{C}_{0,0,0}$, while the dashed arrows are vertices in $\mathscr{C}_{1,0,1} \simeq \mathscr{C}_{1,0,0}$. Thus contracting along the dotted and dashed arrows, we get to the picture of a 3-morphism in an arbitrary tricategory.

Definition 7.73. The *homotopy category* of a d-fold Segal space $\mathscr{C}$ is the (ordinary) category $\mathfrak{h}_1\mathscr{C}$, which has as objects the vertices of $\mathscr{C}_{\mathbf{0}}$ for $\mathbf{0} \in \Delta^{\times d}$. For each $x, y \in \mathscr{C}_{\mathbf{0}}$, we let

$$\mathscr{C}(x,y) := \Delta^0 \times^x_{\mathscr{C}_{\mathbf{0}_1}} \mathscr{C}_{\mathbf{1}_1} \times^y_{\mathscr{C}_{\mathbf{0}_1}} \Delta^0$$

be the $(d-1)$-fold Segal space of morphisms from x to y, where $\mathbf{1}_1, \mathbf{0}_1 \in \Delta^{\{1\}}$ are the evident simplices in the first simplicial direction in the product $\Delta^{\times d}$. The set of morphisms

$$(\mathfrak{h}_1\mathscr{C})(x,y)$$

from x to y is then given as the set of isomorphism classes of objects in $\mathfrak{h}_1(\mathscr{C}(x,y))$, which is already defined by induction. Composition is defined using the Segal condition in the first index.

Definition 7.74. The *homotopy bicategory* $\mathfrak{h}_2\mathscr{C}$ of a d-fold (complete) Segal space $\mathscr{C}$ is the bicategory which has as objects the vertices of $\mathscr{C}_{\mathbf{0}}$ for $\mathbf{0} \in \Delta^{\times d}$ and for $x, y \in \mathscr{C}_{\mathbf{0}}$ the hom-category

$$\mathfrak{h}_2\mathscr{C}(x,y) := \mathfrak{h}_1(\mathscr{C}(x,y))$$

is the homotopy 1-category of the $(d-1)$-fold Segal space $\mathscr{C}(x,y)$ defined by

$$\Delta^0 \times^x_{\mathscr{C}_{\mathbf{0}_S}} \mathscr{C}_{\mathbf{1}_S} \times^y_{\mathscr{C}_{\mathbf{0}_S}} \Delta^0$$

where $S = \{1,3,4,\ldots,d\} \subset \{1,2,\ldots,d\}$ and $\mathbf{0}_S \in \Delta^S$ is the 0-multisimplex, while $\mathbf{1}_S \in \Delta^S$ is given by $\mathbf{1}_S(s) = [0]$ if $s \neq 1$ and $\mathbf{1}_S(1) = [1]$. Here Δ^0 denotes the terminal object in $\mathrm{Psh}_\Delta(\Delta^{\times 2})$. Horizontal composition is then again defined by means of the Segal condition in the second argument.

Remark 7.75. More generally, if $\mathscr{C}$ is an (∞, d)-category, then we can define a d-category $\mathfrak{h}_d\mathscr{C}$ as follows:

- For $k < d$, the k-morphisms of $\mathfrak{h}_d\mathscr{C}$ are the k-morphisms of $\mathscr{C}$.
- The d-morphisms of $\mathfrak{h}_d\mathscr{C}$ are given by isomorphism classes of d-morphisms in $\mathscr{C}$.

In [24] Lurie states that this construction can be characterized by a universal property: Let $\mathscr{D}$ be a d-category, which we can regard as an (∞, d)-category which has only identity k-morphisms for $k > d$. Then functors (of d-categories) from $\mathfrak{h}_n\mathscr{C}$ to $\mathscr{D}$ can be identified with functors (of (∞, d)-categories) from $\mathscr{C} \to \mathscr{D}$, that is,

$$\mathrm{Hom}(\mathscr{D}, \mathfrak{h}_d\mathscr{C}) \approx \mathbb{R}\mathrm{Hom}(\mathscr{D}, \mathscr{C})$$

Definition 7.76. A morphism $\zeta\colon \mathscr{C} \to \mathscr{D}$ of d-fold (d-uple) Segal spaces is a *Dwyer-Kan equivalence* if

- the induced functor $\mathfrak{h}_1\zeta\colon \mathfrak{h}_1\mathscr{C} \to \mathfrak{h}_1\mathscr{D}$ is essentially surjective.
- for each pair of objects x, y in $\mathscr{C}$, the induced morphism $\mathscr{C}(x, y) \to \mathscr{D}(\zeta x, \zeta y)$ is a Dwyer Kan equivalence of $(d-1)$-fold Segal spaces.

Remark 7.77. The $(\infty, d-1)$-category $\mathscr{C}(x, y)$ in the above definition is called the $(\infty, d-1)$-category of morphisms in $\mathscr{C}$ from x to y.

7.5.2. *Truncation, Extension and Loopings.* Given an (∞, d)-category, for $k \leqslant d$, we may consider its (∞, k)-truncation, or k-truncation, which is the (∞, k)-category obtained by discarding all the non-invertible m-morphisms for $k < m \leqslant d$.

Definition 7.78. The *k-truncation* $\mathfrak{T}_k\colon \mathrm{SeSp}_d \to \mathrm{SeSp}_k$ sends a multisimplicial space $\mathscr{C}$ to

$$\mathfrak{T}_k\mathscr{C} := \mathscr{C}_{\mathbf{0}_{\{k+1,\dots,d\}}}$$

where $\mathbf{0}_{\{k+1,\dots,d\}} \in \Delta^{\{k+1,\dots,d\}}$ is the corresponding zero-multisimplex.

Remark 7.79. If $\mathscr{C}$ is complete, then its k-truncation $\mathfrak{T}_k\mathscr{C}$ is complete.

If we have an (∞, d)-category, then we can always promote this to an $(\infty, d+1)$-category by letting the $(n+1)$-morphisms be only identities.

Definition 7.80. The *extension functor* $\mathfrak{E}\colon \mathrm{SeSp}_d \to \mathrm{SeSp}_{d+1}$ sends a multisimplicial space $\mathscr{C}$ to the mutisimplicial space

$$\mathfrak{E}\mathscr{C} \in \mathrm{Psh}_\Delta(\Delta^{\times(d+1)})$$

that is constant with respect to the new factor in the product $\Delta^{\times(d+1)}$.

Lemma 7.81. *If $\mathscr{C}$ is a complete d-fold Segal space, then $\mathfrak{E}\mathscr{C}$ is a complete $(d+1)$-fold Segal space.*

Proof. See [8]. □

Lemma 7.82. *The extension functor $\mathfrak{E}$ is left adjoint to the d-th truncation functor $\mathfrak{T}_d$, that is, we have a diagram of adjunctions*

$$\mathrm{SeSp}_d \underset{\mathfrak{T}_d}{\overset{\mathfrak{E}}{\rightleftarrows}} \mathrm{SeSp}_{d+1} \quad (\perp)$$

Proof. It suffices to show that we have an adjunction on representables, since $\mathfrak{E}$ is cocontinuous: However, for each multisimplex $\mathbf{n} = (n_1, \dots, n_{d+1}) \in \Delta^{\times d} \times \Delta$ we have

$$\mathrm{Hom}(\mathfrak{E}\,よ\mathbf{n}, \mathscr{C}) \cong \mathscr{C}(n_1, \dots, n_d, 0, n_{d+1}) \cong \mathrm{Hom}(よ\mathbf{n}, \mathfrak{T}_d\mathscr{C})$$

where $よ\colon \Delta^{\times d} \times \Delta \to \mathrm{Psh}_\Delta(\Delta^{\times d})$ is the Yoneda embedding. □

Definition 7.83. Let $\mathscr{C}$ be a d-fold Segal space, and let x be an object in $\mathscr{C}$, that is, x is a vertex in $\mathscr{C}_{\mathbf{0}}$.

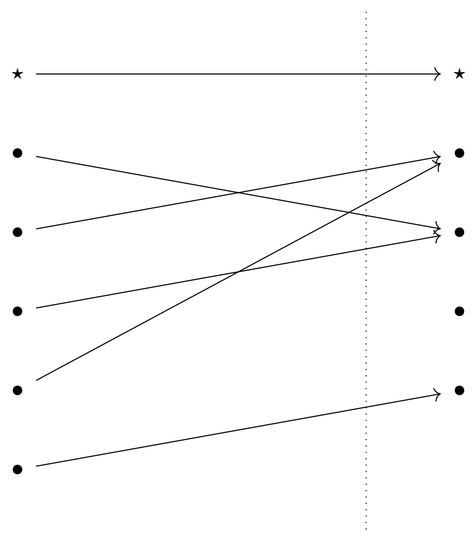
- The *looping* of $\mathscr{C}$ at x is the $(d-1)$-fold Segal space
$$\Omega_x\mathscr{C} := \mathscr{C}(x,x) := \Delta^0 \times^x_{\mathscr{C}_{\mathbf{0}_1}} \mathscr{C}_{\mathbf{1}_1} \times^x_{\mathscr{C}_{\mathbf{0}_1}} \Delta^0$$
- For $1 \leqslant k \leqslant d$, the *$k$-fold iterated looping* of $\mathscr{C}$ at x is the $(d-k)$-fold Segal space
$$\Omega^k_x\mathscr{C} := \Omega_x(\Omega^{k-1}_x\mathscr{C})$$
where we view x as a trivial k-morphism via the degeneracy maps, and $\Omega^0_x\mathscr{C} := \mathscr{C}$.

7.6. **Symmetric Monoidal** (∞, d)**-categories.** When we were concerned with defining (∞, d)-categories, the main idea was to add extra simplicial layers, i.e., every simplicial layer, say the i-th, encoded the notion of a non-trivial space of i-morphisms. On the one hand this was possible since simplicial sets are generalizations of (small) categories (after all we have a fully faithful embedding Cat $\to$ sSet). On the other hand, the combinatorial nature of simplicial sets allowed us to work with these notions rather comfortably. In order to encapsulate the notion of symmetric monoidality a similar machinery will be at play. This time we shall not resort to the simplex category Δ as the underlying source of structure, but rather make use of Segal's Gamma-category Γ:

Definition 7.84. *Segal's Gamma category* Γ is the opposite category of the skeleton of the category of finite pointed sets $\text{Fin}_\star$, which has as objects the finite pointed sets $\langle l\rangle := \{\star, 1, \dots, l\}$ for $l \in \mathbb{N}$ and morphisms are just functions $\langle l\rangle \to \langle k\rangle$ which fix $\star$. In other words,
$$\Gamma := \text{Fin}^{\text{op}}_\star$$

Lemma 7.85. *Any morphism g in* $\text{Fin}_\star$ *may be written as a composition*
$$g = f \circ \sigma$$
where f is a non-decreasing function, while σ is a permutation (bijection).

Proof. Take a morphism as below



and draw the dotted line so far to the right so that there are no more intersections between the arrows to the right of the dotted line. Taking the morphism that results from cutting off to the right of the dotted line yields an order preserving morphism f, while the morphisms resulting from cutting off everything to the left of the dotted arrow results in a permutation σ. □

We shall see that certain kinds of functors $\mathscr{C}\colon \Gamma^{\mathrm{op}} \to \mathrm{Cat}_{(\infty,d)}$ will give the correct notion of *symmetric monoidal* (∞, d)-categories. To this end, we consider the maps

$$\delta_i\colon \langle m\rangle \to \langle 1\rangle, \qquad j \mapsto \delta_{ij}$$

in $\mathrm{Fin}_\star$, where δ_{ij} is the Kronecker-δ with $\delta_{ij} = \star$ for $i \neq j$. Letting j_Γ be the composition

$$\Gamma \xhookrightarrow{\text{よ}_\Gamma} \mathrm{Psh}(\Gamma) \xrightarrow{\pi_\Delta^\star} \mathrm{Psh}_\Delta(\Gamma)$$

we may embed the morphisms δ_i and then make use of the universal property of the coproduct to obtain the dashed arrow

$$\begin{array}{ccc} j\langle m\rangle & \dashleftarrow & \coprod_{j=1}^{m} j\langle 1\rangle \\ {\scriptstyle j\delta_i}\downarrow & & \uparrow \\ j\langle 1\rangle & = & j\langle 1\rangle \end{array}$$

where we have again just written j instead of j_Γ, since there is no danger of ambiguity. Since $\mathrm{Psh}_\Delta(\Gamma)$ is yet again simplicially enriched in the usual manner, we may look at the induced morphism

$$\mathrm{Map}(j\langle m\rangle, \mathscr{C}) \cong \mathscr{C}\langle m\rangle \xrightarrow{(\delta_1^!,\ldots,\delta_m^!)} (\mathscr{C}\langle 1\rangle)^m \cong \mathrm{Map}(\coprod_{j=1}^{m} j\langle 1\rangle, \mathscr{C})$$

Where $\delta_i^! := \mathscr{C}\delta_i$. In particular, for every $l \in \mathbb{N}$ there is a map in $\mathrm{Fin}_\star$ with

$$\varphi := \varphi_l\colon \langle l\rangle \to \langle 1\rangle, \qquad \star \neq j \mapsto 1$$

The induced map $\varphi^! := \mathscr{C}\varphi$ will be responsible to encode multiplication:

$$\mathscr{C}\langle l\rangle \to \mathscr{C}\langle 1\rangle$$

In fact, by means of a left Bousfield localization we will force the maps $\mathscr{C}\langle m\rangle \to (\mathscr{C}\langle 1\rangle)^m$ to be weak equivalences so that the zig-zag of morphisms

$$(\mathscr{C}\langle 1\rangle)^m \xleftarrow{\simeq} \mathscr{C}\langle m\rangle \xrightarrow{\varphi^!} \mathscr{C}\langle 1\rangle$$

will give rise to the notion of thinking of $\mathscr{C}\langle m\rangle$ as the space of m-tuples that may be multiplied, while $\varphi^!$ will be the multiplication operation itself.

Definition 7.86. A *symmetric monoidal* (∞, d)*-category* is a functor $\mathscr{C}\colon \Gamma^{\mathrm{op}} \to \mathrm{Psh}_\Delta(\Delta^{\times d})$ such that

- $\mathscr{C}$ is a fibrant object with respect to the injective model structure on $\mathrm{Psh}_\Delta(\Delta^{\times d} \times \Gamma)_{\mathrm{inj}}$.
- $\mathscr{C}$ is local with respect to all the maps in Definition 7.66 (where we take the tensor product of each of these maps with the identity on $j\langle l\rangle$ for all $l \in \mathbb{N}$).
- *Segal's special* Γ*-condition:* $\mathscr{C}$ is local with respect to all the maps

$$j\mathbf{n} \times \coprod_{i=1}^{l} j\langle 1\rangle \xrightarrow{j\mathbf{n}\times\sigma_l} j\mathbf{n} \times j\langle l\rangle$$

for all $\mathbf{n} \in \Delta^{\times d}$ and $l \in \mathbb{N}$.

If $\mathscr{C}$ is a symmetric monoidal (∞, d)-category, then we will call $\mathscr{C}\langle 1\rangle$ the *(underlying) symmetric monoidal (∞, d)-category.*

Remark 7.87. Let $\mathscr{C}$ be a symmetric monoidal (∞, d)-category. Let us elaborate on what this entity really is about. First of all the partial evaluation $\mathscr{C}\langle l\rangle$ is an (∞, d)-category for all choices $l \in \Gamma$. The statement that $\mathscr{C}$ satisfies Segal's special Γ-condition boils down to the following: Note first that

$$j\mathbf{n} \times \coprod_{i=1}^{l} j\langle 1\rangle \cong \coprod_{i=1}^{l} j(\mathbf{n}, \langle 1\rangle)$$

which follows from cocontinuity of $j\mathbf{n} \times -$ (after all this is a left adjoint) and from the explicit Similarily, $j\mathbf{n} \times j\langle l\rangle \cong j(\mathbf{n}, \langle l\rangle)$. Therefore, applying the Yoneda Lemma, Segal's special Γ-condition amounts to saying that the maps

$$\mathscr{C}(\mathbf{n}, \langle l\rangle) \xrightarrow{\quad\simeq\quad} \mathscr{C}(\mathbf{n}, \langle 1\rangle)^{l}$$

are weak equivalences of simplicial sets for all $\mathbf{n} \in \Delta^{\times d}$ and for all $\langle l\rangle \in \Gamma$. In particular, for $l = 0$ we have $\langle 0\rangle := \langle \star\rangle$, and therefore we get a weak equivalence

$$\mathrm{Map}(\varnothing, \mathscr{C}) \cong \star \xrightarrow{\quad\simeq\quad} \mathscr{C}(\mathbf{n}, \langle \star\rangle)$$

since $\coprod_{i=1}^{0} j\langle 1\rangle := \varnothing$, the initial simplicial presheaf. This forces all spaces $\mathscr{C}(\mathbf{n}, \langle \star\rangle)$ to be contractible.

Theorem 7.88. *There are model structures* $\mathrm{Cat}_{\infty,d}^{\otimes,\mathrm{glob}}$ *and* $\mathrm{Cat}_{\infty,d}^{\otimes,\mathrm{uple}}$ *on the category of simplicial presheaves* $\mathrm{Psh}_{\Delta}(\Delta^{\times d} \times \Gamma)$ *in which the fibrant objects are precisely the symmetric monoidal d-fold complete Segal spaces and the symmetric monoidal d-uple complete Segal spaces, respectively. Both these model structures are defined by means of the corresponding left Bousfield localizations.*

Proposition 7.89. *Any symmetric monoidal (∞, d)-category induces a symmetric monoidal category, if we pass to the corresponding homotopy* 1*-category.*

Proof Sketch. For simplicity let $d = 1$. Fix a symmetric monoidal ∞-category $\mathscr{C} \in \mathrm{Cat}_{(\infty,1)}^{\otimes}$. By definition, we know that the induced morphism

$$\mathscr{C}\langle 2\rangle \xrightarrow{\simeq} \mathscr{C}\langle 1\rangle \times \mathscr{C}\langle 1\rangle$$

is a weak equivalence. By Whitehead's Theorem 5.28, since $\mathscr{C}\langle 2\rangle$ and $\mathscr{C}\langle 1\rangle \times \mathscr{C}\langle 1\rangle$ are bifibrant in $\mathrm{Cat}_{(\infty,1)}$, the weak equivalence $(\delta_1^!, \delta_2^!) := (\mathscr{C}\delta_1, \mathscr{C}\delta_2)$ has a homotopy inverse $m\colon \mathscr{C}\langle 1\rangle \times \mathscr{C}\langle 1\rangle \to \mathscr{C}\langle 2\rangle$, so we may define a tensor ∞-functor as the composition

$$\begin{array}{ccc} \mathscr{C}\langle 1\rangle \times \mathscr{C}\langle 1\rangle & \xrightarrow{\quad m \quad} & \mathscr{C}\langle 2\rangle \\ & {\scriptstyle \otimes} \searrow \quad \swarrow {\scriptstyle \varphi^!} & \\ & \mathscr{C}\langle 1\rangle & \end{array}$$

From the twist isomorphism

$$t\colon \langle 2\rangle \to \langle 2\rangle, \qquad 1,2 \mapsto 2,1$$

we obtain an isomorphism $t^! := \mathscr{C}(t)\colon \mathscr{C}\langle 2\rangle \to \mathscr{C}\langle 2\rangle$. This isomorphism induces a map τ given by the composition

$$\mathscr{C}\langle 1\rangle \times \mathscr{C}\langle 1\rangle \xrightarrow{\quad m \quad} \mathscr{C}\langle 2\rangle \xrightarrow{\quad t^! \quad} \mathscr{C}\langle 2\rangle \xrightarrow{(\delta_1^!,\delta_2^!)} \mathscr{C}\langle 1\rangle \times \mathscr{C}\langle 1\rangle$$

We then realize that if we pass to homotopy categories, the map τ precisely induces the functor

$$\mathfrak{h}_1\mathscr{C} \times \mathfrak{h}_1\mathscr{C} \to \mathfrak{h}_1\mathscr{C} \times \mathfrak{h}_1\mathscr{C}, \qquad (c,c') \mapsto (c',c)$$

which follows from $(\delta_1^!,\delta_2^!) \circ t^! = (\delta_2^!,\delta_1^!)$. In particular, we have

$$\begin{aligned} \otimes \circ \tau &\simeq (\varphi^! m)(\delta_1^!,\delta_2^!)t^! m \\ &\simeq \varphi^! t^! m \\ &= \varphi^! m \\ &= \otimes \end{aligned}$$

and passing to the homotopy 1-category we obtain a natural isomorphism

$$\otimes \circ \tau \cong \otimes$$

where $\tau\colon \mathfrak{h}_1\mathscr{C}\langle 1\rangle \times \mathfrak{h}_1\mathscr{C}\langle 1\rangle \to \mathfrak{h}_1\mathscr{C}\langle 1\rangle$ and $\otimes\colon \mathfrak{h}_1\mathscr{C}\langle 1\rangle \times \mathfrak{h}_1\mathscr{C}\langle 1\rangle \to \mathfrak{h}_1\mathscr{C}\langle 1\rangle$ denote the induced functors on homotopy categories. Analogously, the associator of the tensor product may be given by picking a homotopy inverse of the morphism

$$\mathscr{C}\langle 3\rangle \xrightarrow{\simeq} \mathscr{C}\langle 1\rangle^3$$

Write $a\colon \mathscr{C}\langle 1\rangle^3 \to \mathscr{C}\langle 3\rangle$ for such an homotopy inverse. We may then consider the commutative diagram

$$\begin{array}{ccccc}
\mathscr{C}\langle 3\rangle & \xrightarrow{\quad q^! \quad} & \mathscr{C}\langle 2\rangle & \xrightarrow{\quad \varphi^! \quad} & \mathscr{C}\langle 1\rangle \\
\downarrow{\scriptstyle (f^!,g^!)} & & \downarrow{\scriptstyle (\delta_1^!,\delta_2^!)} & & \\
\mathscr{C}\langle 2\rangle \times \mathscr{C}\langle 1\rangle & \xrightarrow[\varphi^! \times \mathrm{id}]{} & \mathscr{C}\langle 1\rangle \times \mathscr{C}\langle 1\rangle & & \\
\downarrow{\scriptstyle m \times \mathrm{id}} & & & & \\
\mathscr{C}\langle 1\rangle^3 & & & &
\end{array}$$

where $q\colon \langle 3\rangle \to \langle 2\rangle, 1,2,3 \mapsto 1,2,2$ and the morphisms $f\colon \langle 3\rangle \to \langle 2\rangle$ and $g\colon \langle 3\rangle \to \langle 1\rangle$ are given by $1,2,3 \mapsto 1,2,\star$ and $1,2,3 \mapsto \star,\star,1$, respectively.

$$b\colon \langle 3\rangle \to \langle 3\rangle, \qquad 1,2,3 \mapsto 2,3,1$$

Define

$$F := a \circ ((\delta_1^!,\delta_2^!) \times \mathrm{id})\colon \mathscr{C}\langle 2\rangle \times \mathscr{C}\langle 1\rangle \to \mathscr{C}\langle 3\rangle$$

By construction

$$(f^!,g^!)F \simeq \mathrm{id}$$

By making use of the above commutative diagram we obtain a homotopy equivalence

$$(\delta_1^!,\delta_2^!) \circ q^! \circ F = (\varphi^! \times \mathrm{id}) \circ (f^!,g^!) \circ F \simeq \varphi^! \times \mathrm{id}$$

Again by construction

$$(- \otimes -) \otimes - = \varphi^! \circ m \circ (\varphi^! \times \mathrm{id}) \circ (m \times \mathrm{id})$$

and by what we have established before we obtain equivalences

$$\begin{aligned}(-\otimes -)\otimes - &\simeq \varphi^! \circ m \circ (\delta_1^!, \delta_2^!) \circ q^! \circ F \circ (m \times \mathrm{id}) \\ &\simeq \varphi^! \circ q^! \circ F \circ (m \times \mathrm{id}) \\ &= \varphi^! \circ q^! \circ a \circ ((\delta_1^!, \delta_2^!) \times \mathrm{id}) \circ (m \times \mathrm{id}) \\ &\simeq \varphi^! \circ q^! \circ a\end{aligned}$$

Analogously, one shows $-\otimes(-\otimes -) \simeq \varphi^! \circ q^! \circ a$ and therefore we have established

$$(-\otimes -)\otimes - \simeq -\otimes(-\otimes -)$$

Passing to the respective homotopy categories this yields the associator. We then consider the unique morphism $u\colon \langle\star\rangle \to \langle 1\rangle$. From this we obtain a morphism $u^! := \mathscr{C}(u)\colon \mathscr{C}\langle\star\rangle \to \mathscr{C}\langle 1\rangle$. Since, by assumption, the homotopy categeory $\mathfrak{h}_1(\mathscr{C}\langle\star\rangle)$ is the terminal category, the induced functor on homotopy categories simply picks out an object $\mathbb{1} \in \mathfrak{h}_1(\mathscr{C}\langle 1\rangle)$ In order to verify that this object behaves like a monoidal unit, let $\iota_i\colon \langle 1\rangle \to \langle 2\rangle$ be the unique morphism in $\mathrm{Fin}_\star$ which sends 1 to $i \in \{1,2\}$. This induces morphisms $\iota_i^! := \mathscr{C}(\iota_i)\colon \mathscr{C}\langle 1\rangle \to \mathscr{C}\langle 2\rangle$ and these may be identified with

$$\iota_1^!\colon x \mapsto (x, \mathbb{1}), \qquad \iota_2^!\colon x \mapsto (\mathbb{1}, x)$$

Finally, the equalities $\varphi \circ \iota_1 = 1_{\langle 1\rangle} = \varphi \circ \iota_2$ give rise to

$$\varphi^! \circ m \circ (\delta_1^!, \delta_2^!) \circ \iota_i^! \simeq \varphi^! \iota_i^! = \mathrm{id}_{\mathscr{C}\langle 1\rangle}$$

where the LHS is either $\mathbb{1}\otimes -$ or $-\otimes \mathbb{1}$ depending on $i \in \{1,2\}$. Passing to homotopy categories, we obtain natural isomorphisms

$$\rho\colon (-)\otimes \mathbb{1} \xrightarrow{\cong} \mathrm{id}, \qquad \lambda\colon \mathbb{1}\otimes(-) \xrightarrow{\cong} \mathrm{id}$$

in the respective homotopy categories. For more details see [39]. □

Example 7.90. Let $\mathscr{C}$ be a strict symmetric monoidal category (by Maclane's coherence Theorem this is not really a restriction, see the Nlab page coherence theorem for monoidal categories). For $\mathscr{W} := \mathscr{C}^\times$ the maximal subgroupoid of $\mathscr{C}$, the pair $(\mathscr{C}, \mathscr{W})$ gives rise to a homotopical category which is saturated. Therefore, the Rezk nerve

$$\mathfrak{N}^\infty(\mathscr{C}, \mathscr{W})$$

is a complete Segal space. We claim that, by means of the symmetric monoidal structure on $\mathscr{C}$, this may be extended to a symmetric monoidal $(\infty,1)$-category. In fact, we shall define a functor

$$\mathscr{C}\colon \Gamma^{\mathrm{op}} \to \mathrm{Psh}_\Delta(\Delta^{\times d})$$

which will constitute a symmetric monoidal $(\infty,1)$-category. To this end, we note that $\mathscr{W}^m = (\mathscr{C}^m)^\times$. For an object $\langle m\rangle \in \Gamma$ let

$$\mathscr{C}\langle m\rangle := \mathfrak{N}^\infty(\mathscr{C}^m, \mathscr{W}^m)_{\bullet\bullet}$$

Next up, let us see what $\mathscr{C}$ shall do to the multiplication map $\varphi\colon \langle 2\rangle \to \langle 1\rangle, 1,2 \mapsto 1$. Of course, this should induce a map $\mathscr{C}\langle 2\rangle \to \mathscr{C}\langle 1\rangle$. Let us start by considering $\mathscr{C}\langle 2\rangle_{0\bullet}$: Its l-simplices are given by

$$\mathfrak{N}^\infty(\mathscr{C}\times\mathscr{C}, \mathscr{W}\times\mathscr{W})_{0,l} = \mathfrak{N}(\mathrm{we}((\mathscr{C}\times\mathscr{C})^{[0]})_l = \mathfrak{N}(\mathscr{W}\times\mathscr{W})_l \cong \mathfrak{N}\mathscr{W}_l \times \mathfrak{N}\mathscr{W}_l$$

Thus an l-simplex in $\mathscr{C}\langle 2\rangle_{0,\bullet}$ is a pair of two l-tuples of composable isomorphisms

$$C_0 \xrightarrow{\in\mathscr{W}} C_1 \xrightarrow{\in\mathscr{W}} \dots \xrightarrow{\in\mathscr{W}} C_l$$

$$D_0 \xrightarrow{\in\mathscr{W}} D_1 \xrightarrow{\in\mathscr{W}} \dots \xrightarrow{\in\mathscr{W}} D_l$$

We may then use the symmetric monoidal structure on $\mathscr{C}$ to map this pair of l-tuples to

$$C_0 \otimes D_0 \xrightarrow{\in\mathscr{W}} C_1 \otimes D_1 \xrightarrow{\in\mathscr{W}} \dots \xrightarrow{\in\mathscr{W}} C_l \otimes D_l$$

More generally, an l-simplex in

$$\mathscr{C}\langle 2\rangle_{k\bullet} = \mathfrak{N}(\mathrm{we}(\mathscr{C} \times \mathscr{C})^{[k]})_\bullet$$

is a pair of diagrams

$$\begin{array}{ccccccc}
C_{0,0} & \longrightarrow & C_{1,0} & \longrightarrow & \dots & \longrightarrow & C_{k,0} \\
\mathscr{W}\ni\downarrow & & \mathscr{W}\ni\downarrow & & & & \downarrow\in\mathscr{W} \\
C_{0,1} & \longrightarrow & C_{1,1} & \longrightarrow & \dots & \longrightarrow & C_{k,1} \\
\mathscr{W}\ni\downarrow & & \mathscr{W}\ni\downarrow & & & & \downarrow\in\mathscr{W} \\
\vdots & & \vdots & & & & \vdots \\
\mathscr{W}\ni\downarrow & & \mathscr{W}\ni\downarrow & & & & \downarrow\in\mathscr{W} \\
C_{0,l} & \longrightarrow & C_{1,l} & \longrightarrow & \dots & \longrightarrow & C_{k,l}
\end{array}
\qquad
\begin{array}{ccccccc}
D_{0,0} & \longrightarrow & D_{1,0} & \longrightarrow & \dots & \longrightarrow & D_{k,0} \\
\mathscr{W}\ni\downarrow & & \mathscr{W}\ni\downarrow & & & & \downarrow\in\mathscr{W} \\
D_{0,1} & \longrightarrow & D_{1,1} & \longrightarrow & \dots & \longrightarrow & D_{k,1} \\
\mathscr{W}\ni\downarrow & & \mathscr{W}\ni\downarrow & & & & \downarrow\in\mathscr{W} \\
\vdots & & \vdots & & & & \vdots \\
\mathscr{W}\ni\downarrow & & \mathscr{W}\ni\downarrow & & & & \downarrow\in\mathscr{W} \\
D_{0,l} & \longrightarrow & D_{1,l} & \longrightarrow & \dots & \longrightarrow & D_{k,l}
\end{array}$$

which shall be sent to the diagram

$$\begin{array}{ccccccc}
C_{0,0}\otimes D_{0,0} & \longrightarrow & C_{1,0}\otimes D_{1,0} & \longrightarrow & \dots & \longrightarrow & C_{k,0}\otimes D_{k,0} \\
\mathscr{W}\ni\downarrow & & \mathscr{W}\ni\downarrow & & & & \downarrow\in\mathscr{W} \\
C_{0,1}\otimes D_{0,1} & \longrightarrow & C_{1,1}\otimes D_{1,1} & \longrightarrow & \dots & \longrightarrow & C_{k,1}\otimes D_{k,1} \\
\mathscr{W}\ni\downarrow & & \mathscr{W}\ni\downarrow & & & & \downarrow\in\mathscr{W} \\
\vdots & & \vdots & & & & \vdots \\
\mathscr{W}\ni\downarrow & & \mathscr{W}\ni\downarrow & & & & \downarrow\in\mathscr{W} \\
C_{0,l}\otimes D_{0,1} & \longrightarrow & C_{1,l}\otimes D_{1,1} & \longrightarrow & \dots & \longrightarrow & C_{k,l}\otimes D_{k,l}
\end{array}$$

More generally, a morphism $\langle m\rangle \to \langle m'\rangle$ induces a map $\mathscr{C}\langle m\rangle \to \mathscr{C}\langle m'\rangle$ in the following way: By Lemma 7.85 it is enough to define this assignment for permutations and order preserving functions. We can interpret the $(\infty,1)$-category $\mathscr{C}\langle m\rangle$ as a space of m-tuples of commutative grids, or commutative diagrams. Likewise, $\mathscr{C}\langle m'\rangle$ is the space of m'-tuples of commutative grids. A permutation $\sigma\colon \langle m\rangle \to \langle m\rangle$ induces the map $\mathscr{C}\langle m\rangle \to \mathscr{C}\langle m\rangle$ which takes an m-tuple of grids switches their ordering according to σ and thus yields a permuted m-tuple of grids. For an order preserving function $f\colon \langle m\rangle \to \langle m'\rangle$ we obtain a morphism $\mathscr{C}\langle m\rangle \to \mathscr{C}\langle m'\rangle$ which takes an m-tuple of grids and maps it to the m'-tule of grids by looking at the preimages $f^{-1}(j)$ for $j \in \langle m\rangle$. In fact, all the information of $f^{-1}(\star)$ is discarded, and for any set $f^{-1}(j) = \{i_1, \dots, i_s\}$ one takes the tensor product over all the grids numbered by i_k (in the example above, we take the tensor product of the first grid with the second one). All that is left to check is that Segal's special Γ condition is satisfied. However, this follows from

$$\mathscr{C}\langle m\rangle_{k,\bullet} = \mathfrak{N}^\infty(\mathscr{C}^m, \mathscr{W}^m)_{k,\bullet}$$

$$
\begin{aligned}
&= \mathfrak{N}(\mathrm{we}((\mathscr{C}^m)^{[k]}))_\bullet \\
&\cong \mathfrak{N}(\mathrm{we}(\mathscr{C}^{[k]})^m)_\bullet \\
&\cong \mathfrak{N}(\mathrm{we}(\mathscr{C}^{[k]}))_\bullet^m \\
&= (\mathscr{C}\langle 1\rangle_{k,\bullet})^m
\end{aligned}
$$

and from the definition of the maps $\mathscr{C}\delta_i$ (an m-tuple of grids is mapped to the i-th grid). The above construction will be denoted by

$$\mathfrak{N}_\otimes^\infty \mathscr{C} \in \mathrm{Psh}_\Delta(\Delta \times \mathit{\Gamma})$$

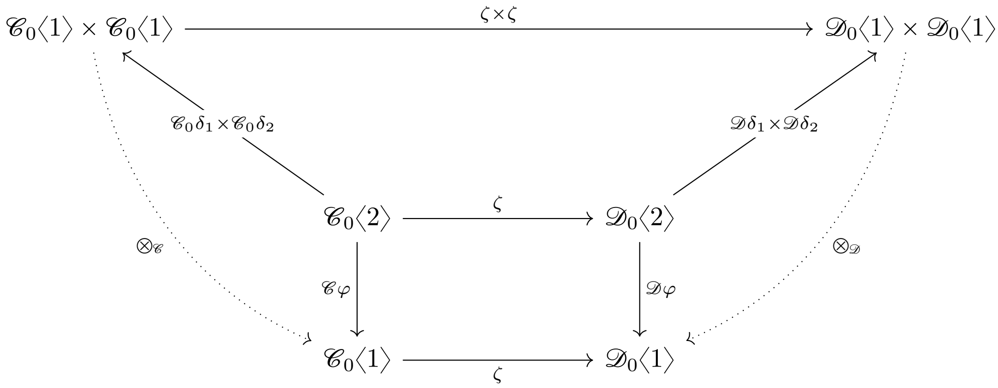
Definition 7.91. Let $\mathscr{C}$ and $\mathscr{D}$ be symmetric monoidal (∞, d)-categories.

- A *symmetric monoidal* ∞*-functor* (or *symmetric monoidal* (∞, d)*-functor*) is a natural transformation $\mathscr{C} \to \mathscr{D}$.
- A *symmetric monoidal* ∞*-natural transformation* is a homotopy $h\colon \mathscr{C} \times j_{\Delta,1}[1] \to \mathscr{D}$.

Remark 7.92. Let $\mathscr{C}, \mathscr{D} \in \mathrm{Cat}^\otimes_{(\infty,1)}$ be fibrant, and let $\mathscr{C} \xrightarrow{\zeta} \mathscr{D}$ be a symmetric monoidal ∞-functor. Then ζ determines morphisms on objects and 1-morphisms:

$$\zeta([0],\langle 1\rangle)\colon \mathscr{C}([0],\langle 1\rangle) \to \mathscr{D}([0],\langle 1\rangle), \qquad \zeta([1],\langle 1\rangle)\colon \mathscr{C}([1],\langle 1\rangle) \to \mathscr{D}([1],\langle 1\rangle)$$

which we will also simply denote by ζ for simplicity. We then observe that commutativity of the diagram



yields compatibility of ζ with the tensor products in $\mathscr{C}$ and $\mathscr{D}$, respectively. In other words, $\zeta(x) \otimes_{\mathscr{D}} \zeta(y) \simeq \zeta(x \otimes_{\mathscr{C}} y)$. Analogously, one verifies the other properties a symmetric monoidal functor has (up to homotopy), which establishes that the induced functor $\mathfrak{h}_1\mathscr{C}\langle 1\rangle \to \mathfrak{h}_1\mathscr{D}\langle 1\rangle$ is a symmetric monoidal functor.

7.7. **Smooth symmetric monoidal (∞, d)-categories.** The next notion we want to include is that of *smooth* ∞-categories. Very roughly speaking, a *smooth* (∞, d)*-category* is an ∞-sheaf of (∞, d)-categories. In other words, local information of (∞, d)-categories may be glued to yield a global (∞, d)-category for all good covers.

Definition 7.93. The *category of cartesian spaces*, denoted Cart, has as objects sets U for which there exists a natural number $n \in \mathbb{N}$ such that U is an open subset of $\mathbb{R}^n$ and U is (smoothly) diffeomorphic to $\mathbb{R}^n$. Morphisms in Cart are just smooth maps.

We may yet again define a map j_{Cart} as the composition

$$\mathrm{Cart} \xhookrightarrow{\;よ\;} \mathrm{Psh}(\mathrm{Cart}) \longrightarrow \mathrm{Psh}_\Delta(\mathrm{Cart}) \longrightarrow \mathrm{Psh}_\Delta(\Delta^{\times d} \times \mathit{\Gamma} \times \mathrm{Cart})$$

and again we just write j, if there is no danger of ambiguity. Moreover, we also extend all the codomains of the maps $j_{\mathit{\Gamma}}, j_{\Delta^A}, j_{\Delta,k}$ and so on to $\mathrm{Psh}_\Delta(\Delta^{\times d} \times \mathit{\Gamma} \times$

Cart). The category Cart may be turned into a site by equipping it with the coverage of good open covers, i.e., open covers for which every finite intersection is either empty or diffeomorphic to $\mathbb{R}^n$ for some n. In this setting we recall the definition of the *Čech nerve*:

Definition 7.94. For $\mathscr{U} := \{U_i\}_{i\in\mathscr{I}}$ a good open cover of V in Cart, the *Čech nerve* $\mathbf{C}\mathscr{U} \in \mathrm{Psh}_\Delta(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})$ has as its m-simplices the presheaf

$$\coprod_{\zeta\colon m} jU_\zeta$$

where $\zeta\colon m$ should mean that ζ runs over all those $(m+1)$-tuples $(\zeta_0, \ldots, \zeta_m) \in \mathscr{I}^{m+1}$ for which

$$U_\zeta := \bigcap_{i=0}^{m} U_{\zeta_i} \neq \varnothing$$

We then define the inclusions

$$\iota^k_{\zeta_0\ldots\zeta_{m+1}} \colon U_{(\zeta_0,\ldots,\zeta_{m+1})} \hookrightarrow U_{(\zeta_0,\ldots,\widehat{\zeta_k},\ldots\zeta_{m+1})}$$

where the hat means omission. Now this yields the face maps for the Čech nerve by means of the universal property of the coproduct:

$$\begin{array}{ccc} \coprod_{m+1\colon\zeta} jU_{(\zeta_0,\ldots,\zeta_{m+1})} & \overset{\exists! d_k}{\dashrightarrow} & \coprod_{m\colon\zeta} jU_{(\zeta_0,\ldots,\zeta_m)} \\ \uparrow & & \uparrow \\ jU_{(\zeta'_0,\ldots,\zeta'_{m+1})} & \xrightarrow[j\iota^k_{\zeta'_0\cdots\zeta'_{m+1}}]{} & jU_{(\zeta'_0,\ldots,\widehat{\zeta'_k},\ldots,\zeta'_{m+1})} \end{array}$$

Similarily, we define the degeneracy maps associated with the Čech nerve by:

$$\begin{array}{ccc} \coprod_{\zeta\colon m} jU_{(\zeta_0,\ldots,\zeta_m)} & \overset{\exists! s_k}{\dashrightarrow} & \coprod_{\zeta\colon m+1} jU_{\zeta_0,\ldots,\zeta_{m+1})} \\ \uparrow & & \uparrow \\ jU_{(\zeta'_0,\ldots,\zeta'_m)} & = & jU_{(\zeta'_0,\ldots,\zeta'_{k-1},\zeta'_k,\zeta'_k,\zeta'_{k+1},\ldots,\zeta'_m)} \end{array}$$

Furthermore, the inclusion maps $U_{(\zeta_0,\ldots,\zeta_m)} \hookrightarrow V$ induce a canonical map

$$\mathbf{C}\mathscr{U} \xrightarrow{\xi_{\mathscr{U}}} jV$$

by means of

$$\begin{array}{ccc} \coprod_{\zeta\colon m} jU_{\zeta_0,\ldots,\zeta_m)} & \dashrightarrow & jV \\ \uparrow & \nearrow_{j(U_{(\zeta'_0,\ldots,\zeta'_m)}\hookrightarrow V)} & \\ jU_{(\zeta'_0,\ldots,\zeta'_m)} & & \end{array}$$

Remark 7.95. More efficiently,

$$\mathbf{C}\mathscr{U} := \int^{(k]\in\Delta} j_\Delta([k]) \odot \coprod_{i_0,\ldots,i_k} jU_{(i_0,\ldots,i_k)}$$

where j_Δ is the composition

$$\Delta \xrightarrow{\ \text{よ}\ } \mathrm{sSet} = \mathrm{Psh}_\Delta(\star) \xrightarrow{\ !^*\ } \mathrm{Psh}_\Delta(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})$$

where $! \colon \Delta^{\times d} \times \Gamma \times \mathrm{Cart} \to \star$ is the canonical projection.

Definition 7.96. A *smooth symmetric monoidal d-uple Segal space* is an object $\mathscr{C} \in \mathrm{Psh}_\Delta(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})$ such that

- $\mathscr{C}$ is a fibrant object with respect to the injective model structure on $\mathrm{Psh}_\Delta(\Delta^{\times d} \times \Gamma)_{\mathrm{inj}}$.
- $\mathscr{C}$ is local with respect to all the maps in Definition 7.86 (where we take the tensor product of each of these maps with the identity on all the other factors of the full product $\Delta^{\times d} \times \Gamma \times \mathrm{Cart}$).
- $\mathscr{C}$ is local with respect to all the maps

  $$j(\mathbf{n}, \langle l \rangle) \times \mathbf{C}\mathscr{U} \xrightarrow{j(\mathbf{n}, \langle l \rangle) \times \xi_{\mathscr{U}}} j(\mathbf{n}, \langle l \rangle) \times jV$$

  for all $\mathbf{n} \in \Delta^{\times d}$ and for all $\langle l \rangle \in \Gamma$.

Remark 7.97. The extra smoothness condition boils down to having weak equivalences

$$\mathbb{R}\mathrm{Map}(j(\mathbf{n}, \langle l \rangle, V), \mathscr{C}) \simeq \mathscr{C}(\mathbf{n}, \langle l \rangle, V)_\bullet \xrightarrow{\simeq} \mathbb{R}\mathrm{Map}(j(\mathbf{n}, \langle l \rangle) \times \mathbf{C}\mathscr{U}, \mathscr{C})$$

However, since any simplicial presheaf may be written as a homotopy colimit (over Δ^{op}) of its individual layers, we have

$$\begin{aligned}
\mathbb{R}\mathrm{Map}(j(\mathbf{n}, \langle l \rangle) \times \mathbf{C}\mathscr{U}, \mathscr{C}) &\simeq \mathbb{R}\mathrm{Map}(j(\mathbf{n}, \langle l \rangle) \times \operatorname*{hocolim}_{[n] \in \Delta^{\mathrm{op}}} \mathbf{C}\mathscr{U}_n, \mathscr{C}) \\
&\simeq \operatorname*{holim}_{[n] \in \Delta} \mathbb{R}\mathrm{Map}(j(\mathbf{n}, \langle l \rangle) \times \mathbf{C}\mathscr{U}_n, \mathscr{C}) \\
&\simeq \operatorname*{holim}_{[n] \in \Delta} \prod_{\zeta \colon n} \mathbb{R}\mathrm{Map}(j(\mathbf{n}, \langle l \rangle) \times jU_\zeta, \mathscr{C}) \\
&\simeq \operatorname*{holim}_{n \in \Delta, \zeta \colon n} \mathscr{C}(\mathbf{n}, \langle l \rangle, U_\zeta)
\end{aligned}$$

In particular, $\operatorname*{holim}_{n \in \Delta, \zeta \colon n} \mathscr{C}(\mathbf{n}, \langle l \rangle, U_\zeta)$ is the homotopy limit of the diagram

$$\prod_{i \in I} \mathscr{C}(\mathbf{n}, \langle l \rangle, U_i) \rightrightarrows \prod_{i_0, i_1 \in I} \mathscr{C}(\mathbf{n}, \langle l \rangle, U_{(i_0, i_1)}) \Rrightarrow \prod_{i_0, i_1, i_2 \in I} \mathscr{C}(\mathbf{n}, \langle l \rangle, U_{(i_0, i_1, i_2)}) \quad \cdots$$

Certainly enough, there also exists the notion of a d-fold smooth symmetric monoidal Segal space and so on. In particular, we have:

Theorem 7.98. *There exist model structures* $\mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{uple}}_{(\infty,d)}$ *and* $\mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{glob}}_{(\infty,d)}$*, which both have the same underlying category* $\mathrm{Psh}_\Delta(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})$*, such that their corresponding fibrant objects are smooth symmetric monoidal d-uple and d-fold Segal spaces, respectively. These model categories are obtained by means of the respective left Bousfield localizations.*

Definition 7.99. Fix a presheaf $\mathscr{C} \colon \mathrm{Cart}^{\mathrm{op}} \to \mathrm{Cat}$.

- The *smooth Rezk nerve* of $\mathscr{C}$ is given by

  $$\mathfrak{N}^{\mathscr{C}^\infty}(\mathscr{C}) := \Big[U \mapsto \mathfrak{N}^\infty(\mathscr{C}(U))\Big] \in \mathrm{Psh}_\Delta(\Delta \times \mathrm{Cart})$$

- If the presheaf $\mathscr{C}$ is actually valued in (strict) symmetric monoidal categories, i.e., $\mathscr{C} \colon \mathrm{Cart}^{\mathrm{op}} \to \mathrm{Cat}^\otimes$, then the *symmetric monoidal smooth Rezk nerve* of $\mathscr{C}$ is given by

  $$\mathfrak{N}^{\mathscr{C}^\infty}_\otimes(\mathscr{C}) := \Big[U \mapsto \mathfrak{N}^\infty_\otimes((\mathscr{C}(U))\Big] \in \mathrm{Psh}_\Delta(\Delta \times \Gamma \times \mathrm{Cart})$$

where $\mathfrak{N}^\infty_\otimes$ denotes the *symmetric monoidal Rezk nerve* as given in Example 7.90.

Remark 7.100. For more details on the above constructions, the reader should consult the 2022 version of [6].

Remark 7.101. Note that the above definition suggests a good notion for what a *smooth* 1*-category* could be. Indeed, such an object should be a functor $\mathscr{C}\colon \mathrm{Cart}^{\mathrm{op}} \to \mathrm{Cat}$ such that $\mathfrak{N}^{\mathscr{C}^\infty}(\mathscr{C})$ satisfies the *smoothness* or *descent condition* with regards to the site Cart.

Example 7.102. Consider a Lie group $\mathfrak{X}$. Any such Lie group gives rise to a model for a smooth symmetric monoidal ∞-groupoid $\mathfrak{X}_\infty \in \mathrm{Psh}_\Delta(\Gamma \times \mathrm{Cart})$ given by the assignment

$$(\langle l\rangle, U) \mapsto \mathscr{C}^\infty(U, \mathfrak{X})^l \in \mathrm{Set} \hookrightarrow \mathrm{sSet}$$

In the same style as in the previous chapter we may define the following notions:

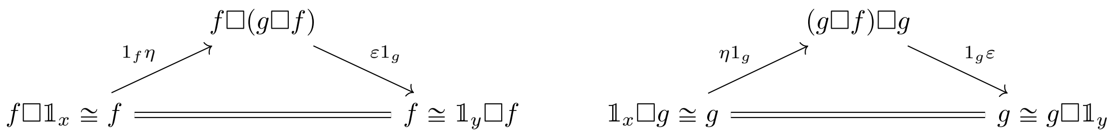
Definition 7.103. Let $\mathscr{C}$ and $\mathscr{D}$ be smooth symmetric monoidal (∞, d)-categories.

- A *smooth symmetric monoidal* ∞*-functor* (or *smooth symmetric monoidal* (∞, d)*-functor*) is a natural transformation $\mathscr{C} \to \mathscr{D}$.
- A *smooth symmetric monoidal* ∞*-natural transformation* is a homotopy $h\colon \mathscr{C} \times j_{\Delta,1}[1] \to \mathscr{D}$.

7.8. **Duals in (∞, d)-Categories.** This chapter is based on [17] and [24].

We shall now try to construct the morphisms in $\mathrm{Psh}_\Delta(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})$ at which we further localize in order to imprint the concept of duals into the very fabric of the mathematical objects that we defined as smooth symmetric monoidal (∞, d)-categories. To this end, we realize that if $\mathscr{B}$ is a bicategory, then it is natural to interpret 1-morphisms as functors and 2-morphisms as natural transformations. In that setting, one can talk about adjunctions in $\mathscr{B}$ in the following sense:

Definition 7.104. Given two composable 1-morphisms $x \xrightarrow{f} y \xrightarrow{g} x$ in a bicategory $\mathscr{B}$ and a 2-morphism $\eta\colon \mathbb{1}_x \to g\Box f$, we call η the *unit of an adjunction*, if there exists $\varepsilon\colon f\Box g \to 1_y$ such that the triangle identities are satisfied:



Example 7.105. Since a monoidal category $\mathscr{C}$ is the same as a bicategory $B\mathscr{C}$ with only one object, a symmetric monoidal category is the same as a bicategory with one object, where composition of 1-morphisms is symmetric. We note that an object $c \in \mathscr{C}$ has a dual $c^\dagger$ if and only if c is right adjoint to $c^\dagger$, when both are viewed as 1-morphisms in $B\mathscr{C}$.

Example 7.106. Let $f\colon x \to y$ be an invertible 1-morphism in a bicategory $\mathscr{B}$. Let g denote its inverse, then we may choose isomorphisms

$$g\Box f \cong \mathbb{1}_x, \qquad f\Box g \cong \mathbb{1}_y$$

which form the unit and counit for an adjunction between f and g. In particular, g is a right adjoint to f, and f is a left adjoint to g. Conversely, any pair of adjoints $f \dashv g$ such that the unit and counit maps $\mathbb{1}_x \to g\Box f$ and $f\Box g \to \mathbb{1}_y$ are isomorphisms exhibit g as an inverse to f, up to isomorphism.

From the previous example we deduce:

Corollary 7.107. *Let $\mathscr{B}$ be a bicategory in which every 2-morphism is invertible, and let f be a 1-morphism in $\mathscr{B}$. Then the following are equivalent:*

- *f is invertible.*
- *f admits a left adjoint.*
- *f admits a right adjoint.*

We recall that any d-fold complete Segal space $\mathscr{C}$ gives rise to its associated *homotopy bicategory* $\mathfrak{h}_2\mathscr{C}$.

Definition 7.108. Let $d > k \geqslant 2$ and fix a smooth symmetric monoidal (∞, d)-category $\mathscr{C}$ and a smooth symmetric monoidal (∞, k)-category $\mathscr{D}$.

- $\mathscr{D}$ is said to *admit adjoints for 1-morphisms*, if its homotopy bicategory $\mathfrak{h}_2\mathscr{D}$ admits adjoints for all 1-morphisms in the sense of Definition 7.104.
- We say that $\mathscr{C}$ *has adjoints for k-morphisms*, if for every fixed $U \in \mathrm{Cart}$ and for all $\mathbf{m} \in \Delta^{\{1,\ldots,k-1\}}$, the $(\infty, d-k+1)$-category $\mathscr{C}(\mathbf{m}, \langle 1\rangle, U)$ has adjoints for 1-morphisms.
- We say that $\mathscr{C}$ *has duals for objects*, if the symmetric monoidal 1-category $\mathfrak{h}_1\mathscr{C}$ has duals for objects.

Remark 7.109. The previous definition may seem incomplete in that we do not consider homotopy coherent adjunctions, however, any adjunction in the homotopy 2-category can be lifted to a homotopy coherent adjunction by [37].

Remark 7.110. By globularity, we realize that the condition that a smooth symmetric monoidal (∞, d)-category $\mathscr{C}$ has adjoints for k-morphisms essentially boils down to saying that the $(\infty, d-k)$-category $\mathscr{D} := \mathscr{C}(\mathbf{1}_{\{1,\ldots,k\}}, \langle 1\rangle, U)$ (where $\mathbf{1}_{\{1,\ldots,k\}} \colon \{1, \ldots, k\} \to \Delta$ is the functor $j \mapsto [1]$) admits adjoints for all 1-morphisms in $\mathfrak{h}_2\mathscr{D}$.

Remark 7.111. The condition that an (∞, d)-category $\mathscr{C}$ has adjoints depends on d. We may always view $\mathscr{C}$ as an $(\infty, d+1)$-category $\mathfrak{C}\mathscr{C}$, in which all $(d+1)$-morphisms are invertible. Yet, $\mathfrak{C}\mathscr{C}$ will not have adjoints for d-morphisms unless $\mathscr{C}$ is an ∞-groupoid.

Corollary 7.112. *Let $\mathscr{C}$ be a smooth symmetric monoidal (∞, d)-category. If every k-morphism in $\mathscr{C}$ is invertible, then $\mathscr{C}$ admits adjoints for k-morphisms.*

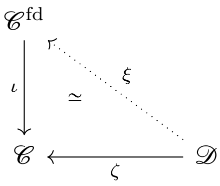
Proof. Follows from Corollary 7.107. □

Remark 7.113. Let $\mathscr{C}$ be a symmetric monoidal (∞, d)-category. We say that an object $c \in \mathscr{C}$ is *invertible* if there is another object $c^{-1} \in \mathscr{C}$ such that the tensor products $c \otimes c^{-1}$ and $c^{-1} \otimes c$ are both isomorphic to the unit object $\mathbb{1} \in \mathscr{C}$. A *Picard ∞-groupoid* is a symmetric monoidal $(\infty, 0)$-category $\mathscr{C}$ such that every object of $\mathscr{C}$ is invertible. By the previous corollary we see that a Picard ∞-groupoid has duals when regarded as an (∞, n)-category for any $n \geqslant 0$.

Claim 7.114 ([24]). Let $\mathscr{C}$ be a symmetric monoidal (∞, d)-category. Then there exists another symmetric monoidal (∞, d)-category $\mathscr{C}^{\mathrm{fd}}$ and a symmetric monoidal functor $\iota \colon \mathscr{C}^{\mathrm{fd}} \to \mathscr{C}$ with the following properties:

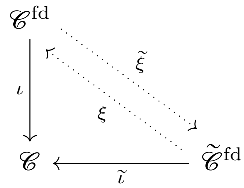
- The symmetric monoidal (∞, d)-category $\mathscr{C}^{\mathrm{fd}}$ has duals.
- For any symmetric monoidal (∞, d)-category $\mathscr{D}$ with duals and any symmetric monoidal functor $\zeta \colon \mathscr{D} \to \mathscr{C}$, there exists a symmetric monoidal functor $\xi \colon \mathscr{D} \to \mathscr{C}^{\mathrm{fd}}$ and an equivalence $\zeta \simeq \iota\xi$. Moreover, ξ is uniquely

determined up to equivalence:



Remark 7.115. Pointing out some subtleties is in order:

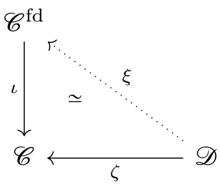
- Let $\mathscr{C}$ be a symmetric monoidal (∞, d)-category, and assume we have two pairs $(\mathscr{C}^{\mathrm{fd}}, \iota)$ and $(\widetilde{\mathscr{C}}^{\mathrm{fd}}, \widetilde{\iota})$ which satisfy the properties of the above claim. By the respective properties we obtain the existence of maps:



  such that $\widetilde{\iota}\widetilde{\xi} \simeq \iota$ and $\iota\xi \simeq \widetilde{\iota}$. Thus in particular,

$$\iota\xi\widetilde{\xi} \simeq \iota, \qquad \widetilde{\iota}\widetilde{\xi}\xi \simeq \widetilde{\iota}$$

  so by uniqueness up to isomorphism, we obtain $\xi\widetilde{\xi} \simeq \mathrm{id}$ and $\widetilde{\xi}\xi \simeq \mathrm{id}$. In summary, $\mathscr{C}^{\mathrm{fd}}$ is uniquely determined up to equivalence.
- For a smooth symmetric monoidal (∞, d)-category, the claim does not change. We consider the same diagrams



  yet every (∞, d)-category involved is now also smooth.
- In the case where $\mathscr{C}$ is a symmetric monoidal $(\infty, 1)$-category we may identify $\mathscr{C}^{\mathrm{fd}}$ with the full subcategory of $\mathscr{C}$ spanned by the dualizable objects in $\mathscr{C}$. More generally, passing from a symmetric monoidal (∞, d)-category to its fully dualizable counterpart $\mathscr{C}^{\mathrm{fd}}$ requires repeatedly discarding objects which do not admit duals and k-morphisms which do not admit left and right adjoints.

Definition 7.116. Let $\mathscr{C}$ be a symmetric monoidal (∞, d)-category. An object $c \in \mathscr{C}$ is called *fully dualizable* if it belongs to the *essential image* of the functor $\iota\colon \mathscr{C}^{\mathrm{fd}} \to \mathscr{C}$.

The goal is now to encode having adjoints for k-morphisms and having duals for objects into a new model category structure $\mathscr{C}^{\infty}\mathrm{Cat}_{\infty,d}^{\otimes,\dagger}$ in which fibrant objects are precisely *smooth symmetric monoidal* (∞, d)*-categories with duals.*

Remark 7.117. In particular, if we have such a model structure, then a trivial fibration

$$\iota\colon \mathscr{C}^{\mathrm{fd}} \to \mathscr{C}$$

in $\mathscr{C}^\infty\mathrm{Cat}_{\infty,d}^{\otimes,\dagger}$ gives rise to the lifting problem

$$\begin{array}{ccc} \varnothing & \hookrightarrow & \mathscr{C}^{\mathrm{fd}} \\ \downarrow & \nearrow & \downarrow \simeq \\ \mathscr{D} & \xrightarrow[\zeta]{} & \mathscr{C} \end{array}$$

for any smooth symmetric monoidal (∞, d)-category with duals $\mathscr{D}$. The existsence of the corresponding lift is assured, since the LHS is a cofibration and the RHS is a trivial fibration, by assumption. In particular, such a lift is unique up to homotopy. The existence of this lift is precisely the content of Claim 7.114 (with the sole difference that the diagram actually strictly commutes).

Definition 7.118. The bicategory Adj is the bicategory freely generated by

- two objects x and y,
- two morphisms $f\colon x \to y$ and $g\colon y \to x$,
- two 2-morphisms $\eta\colon \mathbb{1}_x \to g\Box f$ and $\varepsilon\colon f\Box g \to \mathbb{1}_y$

satisfying the triangle relations:

$$\begin{array}{ccccc} & & f\Box(g\Box f) & & \\ & \overset{1_f\eta}{\nearrow} & & \overset{\varepsilon 1_g}{\searrow} & \\ f\Box\mathbb{1}_x \cong f & & = = = = = & & f \cong \mathbb{1}_y\Box f \end{array} \qquad \begin{array}{ccccc} & & (g\Box f)\Box g & & \\ & \overset{\eta 1_g}{\nearrow} & & \overset{1_g\varepsilon}{\searrow} & \\ \mathbb{1}_x\Box g \cong g & & = = = = = & & g \cong g\Box\mathbb{1}_y \end{array}$$

We call Adj the *free walking adjunction.*

Remark 7.119. Any 2-functor $\mathrm{Adj} \to \mathscr{B}$, for $\mathscr{B}$ a bicategory, uniquely determines an adjunction in $\mathscr{B}$.

We note that any bicategory $\mathscr{B}$ gives rise to a simplicially enriched category $\mathscr{B}_\Delta$, which has as objects the set $\mathscr{B}_0$ and for $x, y \in \mathscr{B}_0$ we have a simplicial mapping object

$$\mathscr{B}_\Delta(x, y) := \mathfrak{N}(\mathscr{B}(x, y)) \in \mathrm{sSet}$$

which is just the standard nerve of the respective hom-category of $\mathscr{B}$. In fact, this determines a functor

$$(-)_\Delta\colon \mathrm{Bicat} \to \mathrm{sSet}\text{-}\mathrm{Cat}$$

between the category of bicategories and the category of simplicially enriched categories. Taking this one step further, there is a canonical functor $\mathfrak{N}_\Delta$ given by the composition

$$\mathrm{sSet}\text{-}\mathrm{Cat} \longrightarrow \mathrm{Cat}^{\Delta^{\mathrm{op}}} \longrightarrow \mathrm{Psh}(\Delta^{\times 2}) \longrightarrow \mathrm{Psh}_\Delta(\Delta^{\times 2})$$

The arrow to the outermost right in the above composition is just interpreting a bisimplicial set as a bisimplicial space by identifying it as a constant bisimplicial space in the new simplicial direction. The morphism $\mathrm{Cat}^{\Delta^{\mathrm{op}}} \to \mathrm{Psh}(\Delta^{\times 2})$ takes an object in $\mathrm{Cat}^{\Delta^{\mathrm{op}}}$ to its levelwise nerve. Finally, the morphism $\mathrm{sSet}\text{-}\mathrm{Cat} \to \mathrm{Cat}^{\mathrm{op}}$ takes a simplicially enriched category $\mathscr{C}$ and views it as a simplicial object in Cat as follows: A simplicially enriched category $\mathscr{C}$ gives rise to a functor $\widetilde{\mathscr{C}}\colon \Delta^{\mathrm{op}} \to \mathrm{Cat}$ which assigns to $[n] \in \Delta$ the category $\widetilde{\mathscr{C}}_n$ which has the same objects as $\mathscr{C}$ and morphisms from objects x to y are given by the set of n-simplices $\mathscr{C}(x, y)_n$. In other words, a simplicially enriched category is the same as a simplicial object in Cat which is constant on objects. Summarizing all this, we get the following definition:

Definition 7.120. The *double nerve* is the functor

$$\mathfrak{N}_2 \colon \mathrm{Bicat} \to \mathrm{Psh}_\Delta(\Delta^{\times 2})$$

given by taking the composition

$$\mathrm{Bicat} \xrightarrow{\ (-)_\Delta\ } \text{sSet-Cat} \xrightarrow{\ \mathfrak{N}_\Delta\ } \mathrm{Psh}_\Delta(\Delta^{\times 2})$$

Having collected these notions, we can embed the walking adjunction into the category of bisimplicial spaces:

Definition 7.121. Write $\mathrm{sub}(f), \mathrm{sub}(g), \mathrm{sub}(\eta), \mathrm{sub}(\varepsilon) \subset \mathrm{Adj}$ for the sub-bicategories generated by $\{f\}, \{g\}, \{f, g, \eta\}$ and $\{f, g, \varepsilon\}$, respectively. In particular, we let

- $\underline{f} := \mathfrak{N}_2\mathrm{sub}(f)$,
- $\underline{g} := \mathfrak{N}_2\mathrm{sub}(g)$,
- $\underline{\eta} := \mathfrak{N}_2\mathrm{sub}(\eta)$,
- $\underline{\varepsilon} := \mathfrak{N}_2\mathrm{sub}(\varepsilon)$,
- $\underline{\mathrm{Adj}} := \mathfrak{N}_2\mathrm{Adj}$.

Definition 7.122. Let $d \geqslant 2, 1 \leqslant k \leqslant d-1$, and $\mathbf{m} \in \Delta^{\{1,\dots,k-1\}}$. Let $p_2 \colon \Delta^{\{k,\dots,d\}} \to \Delta^{\times 2}$ denote the projection onto the first two factors of $\Delta^{\{k,\dots,d\}}$. Consider the functor

$$j\mathbf{m} \odot p_2^\star \colon \mathrm{Psh}_\Delta(\Delta^{\times 2}) \to \mathrm{Psh}_\Delta(\Delta^{\times d})$$

which takes an object $X \in \mathrm{Psh}_\Delta(\Delta^{\times 2})$ to the mutisimplicial space

$$\coprod_{j\mathbf{m}} p_2^\star X, \qquad \Delta^{\{1,\dots,k-1\}} \times \Delta^{\{k,\dots,d\}} \ni (\mathbf{l}, \mathbf{k}) \mapsto \coprod_{\mathrm{Hom}(\mathbf{l},\mathbf{m})} X_{p_2(\mathbf{k}),\bullet} \in \mathrm{sSet}$$

Applying the functor $j\mathbf{m} \odot p_2^\star$ to all the bisimplicial spaces $\underline{f}, \underline{g}, \underline{\eta}, \underline{\varepsilon}$ yields multisimplicial spaces

$$f_\mathbf{m}, g_\mathbf{m}, \eta_\mathbf{m}, \varepsilon_\mathbf{m}, \mathrm{Adj}_\mathbf{m}$$

For d, k and $\mathbf{m}$ as above, we realize that the inclusion 2-functor $\mathrm{sub}(f) \hookrightarrow \mathrm{Adj}$ induces a morphism

$$f_\mathbf{m} \to \mathrm{Adj}_\mathbf{m}$$

In particular, if $\mathscr{F}^\otimes(\star)$ is the free symmetric monoidal category with duals on a single object $\star$, then we may interpret $\mathscr{F}^\otimes(\star)$ as a functor $\Gamma^{\mathrm{op}} \to \mathrm{Cat}$ by

$$\mathscr{F}^\otimes(\star)(\langle l \rangle) := \mathscr{F}^\otimes(\star)^l$$

Taking nerves levelwise we obtain a presheaf on $\Delta \times \Gamma$, which we may promote to a simplicial presheaf on $\Delta \times \Gamma$. Pulling this back once again, we obtain a simplicial presheaf on $\Delta^{\times d} \times \Gamma$, which we denote by $\mathrm{Dual}_\otimes$. Consider the subobject $\dagger \subset \mathrm{Dual}_\otimes$ generated by the image of the object $\star$ inside $\mathrm{Dual}_\otimes$. The inclusion $\star \hookrightarrow \mathscr{F}^\otimes(\star)$ induces a map

$$\dagger \to \mathrm{Dual}_\otimes$$

Definition 7.123. The model category $\mathscr{C}^\infty\mathrm{Cat}_{\infty,d}^{\otimes,\dagger}$ is given by the left Bousfield localization of $\mathscr{C}^\infty\mathrm{Cat}_{\infty,d}^{\otimes,\mathrm{glob}}$ at the morphisms

$$j(\langle l \rangle, V) \times f_\mathbf{m} \longrightarrow j(\langle l \rangle, V) \times \mathrm{Adj}_\mathbf{m}$$

$$j(\langle l \rangle, V) \times \dagger \longrightarrow j(\langle l \rangle, V) \times \mathrm{Dual}_\otimes$$

for all $\langle l \rangle \in \Gamma, V \in \mathrm{Cart}, \mathbf{m} \in \Delta^{\{1,\dots,k-1\}}$. A fibrant object in $\mathscr{C}^\infty\mathrm{Cat}_{\infty,d}^{\otimes,\dagger}$ will be referred to as a *smooth symmetric monoidal* (∞, d)*-category with duals.*

Proposition 7.124. *Let $\mathscr{C}$ be a fibrant object in $\mathscr{C}^\infty\mathrm{Cat}^{\dagger}_{\infty,d}$. Then for all $(\langle l\rangle, V) \in \Gamma \times \mathrm{Cart}$, the (∞,d)-category $\mathscr{C}(\langle l\rangle, V)$ admits duals for all k-morphisms with $1 \leqslant k < d$. In particular, for all $V \in \mathrm{Cart}$, the symmetric monoidal (∞,d)-category $\mathscr{C}(V)$ admits duals for objects.*

Proof Sketch. We shall only verify that $\mathscr{C}$ has duals for objects. The other part of the proof may be found in [17] Proposition 2.3.13. First, since our model structure is simplicial and the map $\dagger \hookrightarrow \mathrm{Dual}_{\otimes}$ is a trivial cofibration, the induced map on simplcial mapping spaces $\mathrm{Map}(\mathrm{Dual}_{\otimes}, \mathscr{C}) \to \mathrm{Map}(\dagger, \mathscr{C})$ is a trivial fibration. In particular, this map is surjective on vertices. We then note that a map $\dagger \to \mathscr{C}$ just picks an object $x \in \mathscr{C}\langle 1\rangle$. On the other hand, a map $\mathrm{Dual}_{\otimes} \to \mathscr{C}$ picks out an object $x^\dagger$ together with unit and counit maps, which witness $x^\dagger$ as the dual of x in the respective homotopy categories. Hence, in total, the existence of a lift for the diagram

$$\begin{array}{ccc} \varnothing & \longrightarrow & \mathrm{Map}(\mathrm{Dual}_{\otimes}, \mathscr{C}) \\ \big\downarrow & \overset{(x,x^\dagger,\eta,\varepsilon)}{\nearrow} & \big\downarrow \simeq \\ j\langle 1\rangle & \xrightarrow[x]{} & \mathrm{Map}(\dagger, \mathscr{C}) \end{array}$$

yields the claim. □

7.9. **Smooth ∞-Functor Categories.** Our goal in this chapter is to define a suitable notion of *smooth symmetric monoidal (∞,d)-functor categories*. From Example 4.50 we recall that the category of simplicial presheaves on a small category $\mathscr{C}$ is always powered, tensored, and enriched over sSet. With these ingredients, the injective model structure $\mathrm{Psh}_{\Delta}(\mathscr{C})_{\mathrm{inj}}$ is a simplicial model category. The key to finding the correct functor ∞-categories is to find a good closed symmetric monoidal structure on $\mathrm{Psh}_{\Delta}(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})$, and then hope that this will behave well with the associated model structures, that is, we want that the corresponding internal hom yields a right Quillen bifunctor.

Notation 7.125. We recall the following notation:

- Given $X, Y \in \mathrm{Psh}_{\Delta}(\mathscr{C})$, for $\mathscr{A}$ a symmetric monoidal category, we denote the simplicial enrichment by
$$\mathrm{Map}(X, Y) \in \mathrm{sSet}$$
- In the special case where $\mathscr{A} = \Delta^{\times d} \times \Gamma \times \mathrm{Cart}$, we endow $\mathrm{Psh}_{\Delta}(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})$ with the symmetric monoidal structure induced by Day convolution, analogously to how we did it in Example 4.56. For $X, Y \in \mathrm{Psh}_{\Delta}(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})$ we will denote the corresponding Day convolution tensor product by
$$X \otimes Y \in \mathrm{Psh}_{\Delta}(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})$$
The Day internal hom will be denoted by
$$\mathrm{Fun}^{\otimes}(X, Y) \in \mathrm{Psh}_{\Delta}(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})$$
We note that this internal hom also has a neat formula:
$$\mathrm{Fun}^{\otimes}(X, Y) = \mathrm{Hom}(X \otimes よ, Y)$$
where $よ : (\Delta^{\times d} \times \Gamma \times \mathrm{Cart})^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{Set}$ is the Yoneda embedding and $\mathrm{Hom}(-,-)$ denotes the set-valued hom-functor of the category $\mathrm{Psh}_{\Delta}(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})$.

According to [16] the *multiple injective model structure* $\mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{uple}}_{(\infty,d)}$ is a symmetric monoidal model category (where the tensor product is Day convolution), which allows for an easy way to define the appropriate homotopical internal hom in the uple case of smooth symmetric monoidal (∞,d)-categories:

Definition 7.126. Let $d \geqslant 0$ and fix arbitrary objects $X, Y \in \mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{uple}}_{(\infty,d)}$. The *homotopical internal hom* in $\mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{uple}}_{(\infty,d)}$ from X to Y, is the derived internal hom:

$$\mathrm{Fun}^{\otimes}_{\mathrm{uple}}(X,Y) := \mathrm{Fun}^{\otimes}(X, R_{\mathrm{uple}}(Y)) \in \mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{uple}}_{(\infty,d)}$$

where R_{uple} denotes a fibrant replacement functor in $\mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{uple}}_{(\infty,d)}$.

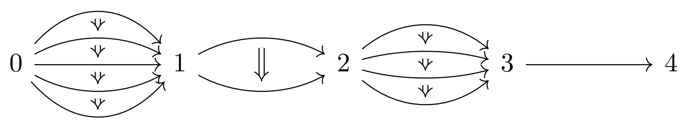
Unfortunately, the globular injective model structure $\mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{glob}}_{(\infty,d)}$ does not satisfy the pushout product axiom, so we cannot compute homotopical internal homs by computing the corresponding derived internal hom (the globularity condition itself is at fault here). The solution to the problem is to transfer the derived internal hom of some other Quillen equivalent model category to our setting. For this, we first have to introduce *Rezk's* Θ_d*-spaces*, which form a cartesian model category:

7.9.1. *Rezk's* Θ_d*-spaces.* The following short exposition is based on [32] and the most recent version of [16].

A Θ_d-space is a simplicial presheaf on *Joyal's category* Θ_d, that is, an object in $\mathrm{Psh}_\Delta(\Theta_d)$. The categories Θ_d for $d \in \mathbb{N}$ will be explained first: We regard Θ_d as a full subcategory of St-d-Cat of strict d-categories (recall that a strict d-category is a category enriched over the category of strict $(d-1)$-categories, see also Example 4.23). The category Θ_0 is the full subcategory of St-0-Cat = Set consisting of the terminal object. The category Θ_1 is the full subcategory of St-1-Cat consisting of the objects $[n]$ for $n \in \mathbb{N}$, where $[n]$ represents the free strict 1-category on the diagram

$$0 \longrightarrow 1 \longrightarrow \dots \longrightarrow n-1 \longrightarrow n$$

In particular, $\Theta_1 = \Delta$, the standard simplex category. The category Θ_2 is the full subcategory of St-2-Cat consisting of objects which are denoted $[m]([n_1], \dots, [n_m])$ for $m, n_1, \dots, n_m \in \mathbb{N}$. This represents the strict 2-category $\mathbf{C}$ which is *"freely generated"* by the objects $\{0, 1, \dots, m\}$, and morphism categories $\mathbf{C}(i-1,i) = [n]_i$. For example, the strict 2-category $[4]([5],[1],[3],[0])$ corresponds to the free 2-category:



More generally, the objects of Θ_d are of the form $[m](\vartheta_1, \dots, \vartheta_m)$, where $m \in \mathbb{N}$ and ϑ_j are objects in Θ_{d-1}. This object then corresponds to the strict d-category $\mathbf{C}$ freely generated by objects $\{0, \dots, m\}$, and a strict $(d-1)$-category of morphisms $\mathbf{C}(i-1,i) = \vartheta_i$. Morphisms of Θ_d are just functors between strict d-categories (enriched functors between St-$(d-1)$-Cat-enriched categories.). Just like $\Delta^{\times d}$, the category Θ_d may be used to define the correct notion of *globular* (∞, d)*-categories*. In fact, one considers the injective model structure $\mathrm{Psh}_\Delta(\Theta_d)_{\mathrm{inj}}$ and then one performs left Bousfield localization with respect to a class of morphisms which encodes some form of Segal conditions as well as completeness conditions (for details see [32]). The resulting model category will be denoted by $\mathrm{Psh}_\Delta(\Theta_d)_{\mathrm{loc}}$ and it is called the *Rezk model structure on Θ_d-spaces*. We then observe that there is a functor $f\colon \Delta^{\times d} \to \Theta_d$ given by the composition

$$\Delta^{\times d} \xrightarrow{f_1} \Delta^{\times(d-1)} \times \Theta_1 \xrightarrow{f_2} \Delta^{\times(d-2)} \times \Theta_2 \xrightarrow{f_3} \dots \xrightarrow{f_{d-1}} \Delta \times \Theta_{d-1} \xrightarrow{f_d} \Theta_d$$

where

$$\begin{array}{ccc} \Delta^{(d-i+1)} \times \Theta_{i-1} & \xrightarrow{\quad f_i \quad} & \Delta^{\times(d-i)} \times \Theta_i \\ ([m_1], \dots, [m_{d-i+1}], \vartheta) & \longmapsto & ([m_1], \dots, [m_{d-i}], [m_{d-i+1}](\vartheta, \dots, \vartheta)) \end{array}$$

For more details on this see [5]. Taking left and right Kan extensions along f, yields an adjunction

$$\mathrm{Psh}_\Delta(\Delta^{\times d}) \begin{array}{c} \xleftarrow{f^\#} \\ \perp \\ \xrightarrow{f^\star} \\ \perp \\ \xrightarrow[f_\star]{} \end{array} \mathrm{Psh}_\Delta(\Theta_d)$$

The adjunction $f^\star \dashv f_\star$ is then found to be a Quillen equivalence between the globular model structure on d-fold Segal spaces (that is, $\mathrm{Cat}^{\mathrm{glob}}_{(\infty,d)}$) and the Rezk model structure on Θ_d-spaces (see [5] Corollary 7.3). From that point of view, fibrant objects in $\mathrm{Psh}_\Delta(\Theta_d)_{\mathrm{loc}}$ describe the notion of globular (∞, d)-categories just as well as d-fold complete Segal spaces do. The advantage of the Rezk model structure is however that it is a cartesian closed model structure, i.e., we have a homotopical internal hom. The idea is to transfer this homtotopical internal hom from Θ_d-spaces to d-fold Segal spaces by means of the Quillen equivalence $f^\star \dashv f_\star$. We then note that we may extend the functor f to a functor $\widetilde{f}\colon \Delta^{\times d} \times \Gamma \times \mathrm{Cart} \to \Theta_d \times \Gamma \times \mathrm{Cart}$ by setting $\widetilde{f} := f \times \mathrm{id}_\Gamma \times \mathrm{id}_{\mathrm{Cart}}$. In particular, denote by $\widetilde{f}^\#$ and $\widetilde{f}_\star$ the corresponding left and right Kan extensions along $\widetilde{f}$. In the newest version of [16], we find the following result:

Proposition 7.127. *Let $d \geqslant 0$ and denote by $\mathrm{Fun}^{\otimes}_{\Theta}(-,-)$ the internal hom in $\mathrm{Psh}_\Delta(\Theta_d \times \Gamma \times \mathrm{Cart})$ with respect to the cartesian closed structure. Furthermore, let $R_{\mathrm{inj}}\colon \mathrm{Psh}_\Delta(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})_{\mathrm{inj}} \to \mathrm{Psh}_\Delta(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})_{\mathrm{inj}}$ be a fibrant replacement functor for the injective model structure. We may then consider the bifunctor*

$$\begin{aligned} \mathrm{Psh}_\Delta(\Delta^{\times d} \times \Gamma \times \mathrm{Cart}) \times \mathrm{Psh}_\Delta(\Delta^{\times d} \times \Gamma \times \mathrm{Cart}) &\to \mathrm{Psh}_\Delta(\Delta^{\times d} \times \Gamma \times \mathrm{Cart}) \\ (Y, Z) &\mapsto R_{\mathrm{inj}} \widetilde{f}^\star \mathrm{Fun}^{\otimes}_{\Theta}(\widetilde{f}_\star Y, \widetilde{f}_\star Z) \end{aligned}$$

Then for all $X, Y, Z \in \mathrm{Psh}_\Delta(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})$ with Z fibrant in the globular model structure and X, Y fibrant in the injective model strcture, the object $R_{\mathrm{inj}} \widetilde{f}^\star \mathrm{Fun}^{\otimes}_{\Theta}(\widetilde{f}_\star Y, \widetilde{f}_\star Z)$ is also fibrant in $\mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{glob}}_{(\infty,d)}$ and, moreover, we have a weak equivalence of derived mapping spaces

$$\mathrm{Map}(X \otimes Y, Z) \simeq \mathrm{Map}(X, R_{\mathrm{inj}} \widetilde{f}^\star \mathrm{Fun}^{\otimes}_{\Theta}(\widetilde{f}_\star Y, \widetilde{f}_\star Z))$$

Proof. This will be Proposition 2.4.3 in the updated version of [16]. □

This motivates:

Definition 7.128. Let $d \geqslant 0$. For $X, Y \in \mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{glob}}_{(\infty,d)}$, the corresponding *homotopical internal Hom* is given by

$$\mathrm{Fun}^{\mathrm{glob}}_{\otimes}(X, Y) := R_{\mathrm{inj}} \widetilde{f}^\star \mathrm{Fun}^{\otimes}_{\Theta}(\widetilde{f}_\star R_{\mathrm{glob}} X, \widetilde{f}_\star R_{\mathrm{glob}} Y)$$

where $R_{\mathrm{inj}}, R_{\mathrm{glob}}$ are fibrant replacement functors for the injective and globular model structures, respectively.

Remark 7.129. We note that $\mathrm{Fun}_{\otimes}^{\mathrm{glob}}$ is not the left or right derived functor of $\mathrm{Fun}^{\otimes}$, but the above proposition assures us that this is yet still the correct notion to go with.

7.10. **Cores and Mapping objects of smooth (∞, d)-Categories.** Recall the following:

Notation 7.130. Let $\mathbf{R} := \mathbf{L} \times \mathbf{R}/\mathbf{L}$ be a product of two categories, and let $X \in \mathrm{Psh}_{\Delta}(\mathbf{R})$. For $l \in \mathbf{L}$, we are able to perform partial evaluation of X at l to obtain a simplicial presheaf $Xl \in \mathrm{Psh}_{\Delta}(\mathbf{R}/\mathbf{L})$.

Definition 7.131. Let $\mathbf{M}, \mathbf{L}/\mathbf{M}, \mathbf{R}/\mathbf{L}$ be symmetric monoidal categories and consider the induced symmetric monoidal categories $\mathbf{L} := \mathbf{M} \times \mathbf{L}/\mathbf{M}$, $\mathbf{R} := \mathbf{L} \times \mathbf{R}/L$ and $\mathbf{R}/\mathbf{M} := \mathbf{L}/\mathbf{M} \times \mathbf{R}/\mathbf{L}$. Endow both $\mathrm{Psh}_{\Delta}(\mathbf{L})$ and $\mathrm{Psh}_{\Delta}(\mathbf{R})$ with the Day convolution closed monoidal structure (analogously to Example 4.56).

- The *powering* of $Y \in \mathrm{Psh}_{\Delta}(\mathbf{R})$ by $X \in \mathrm{Psh}_{\Delta}(\mathbf{L})$ is given by
$$\mathfrak{Map}(X, Y) := [p^{\star} X, Y]_{\mathrm{Day}}$$
where $p \colon \mathbf{L} \times \mathbf{R}/\mathbf{L} \to \mathbf{L}$ is the canonical projection functor and $[-,-]_{\mathrm{Day}}$ is the Day convolution internal hom (see Proposition 4.53).
- For $Y \in \mathrm{Psh}_{\Delta}(\mathbf{R})$ and $X \in \mathrm{Psh}_{\Delta}(\mathbf{L})$, the corresponding *mapping object from X to Y* is given by
$$\mathfrak{Map}_{\mathbf{M}}(X, Y) := \int_{m \in \mathbf{M}} \mathfrak{Map}(Xm, Ym) \in \mathrm{Psh}_{\Delta}(\mathbf{R}/\mathbf{M})$$

Remark 7.132. Let us point out some subtleties:

- The above definition will be of particular interest to us whenever $\mathbf{R}$ (and therefore also $\mathbf{L}$) is a subfactor of $\Delta^{\times d} \times \Gamma \times \mathrm{Cart}$. If $\mathrm{Psh}_{\Delta}(\mathbf{R})$ is endowed with the multiple injective model structure obtained by localizing the factors present in $\mathbf{R}$ (analogously to how we did it for $\mathscr{C}^{\infty}\mathrm{Cat}_{(\infty,d)}^{\otimes,\mathrm{uple}}$), denoted $\mathrm{Psh}_{\Delta}(\mathbf{R})_{\text{mult-inj}}$, then the functor $p^{\star}$ is a left Quillen functor. In particular, the powering
$$\mathfrak{Map}(-,-) \colon \mathrm{Psh}_{\Delta}(\mathbf{L})^{\mathrm{op}}_{\text{mult-inj}} \times \mathrm{Psh}_{\Delta}(\mathbf{R})_{\text{mult-inj}} \to \mathrm{Psh}_{\Delta}(\mathbf{R})_{\text{mult-inj}}$$
is a right Quillen bifunctor.
- If $\mathbf{L} = \mathbf{R}$, then $\mathfrak{Map}(X, Y) = [X, Y]_{\mathrm{Day}}$ is the Day internal Hom from X to Y.
- For $\mathbf{M} = \star$, we have
$$\mathfrak{Map}_{\mathbf{M}}(X, Y) = \mathfrak{Map}(X, Y)$$
Indeed, any wedge
$$(Z, \varphi)$$
for the functor $\star^{\mathrm{op}} \times \star \longrightarrow \mathrm{Psh}_{\Delta}(\mathbf{R}), (\star, \star) \mapsto \mathfrak{Map}(X, Y)$ is trivial in the sense that the wedge condition reads
$$\begin{array}{ccccc} & & \mathfrak{Map}(X,Y) & & \\ & \overset{\varphi}{\nearrow} & & \searrow\!\!= & \\ Z & & & & \mathfrak{Map}(X,Y) \\ & \underset{\varphi}{\searrow} & & \nearrow\!\!= & \\ & & \mathfrak{Map}(X,Y) & & \end{array}$$

which gives no more information than saying that φ is a morphism from Z to $\mathfrak{Map}(X,Y)$ in $\mathrm{Psh}_\Delta(\mathbf{R})$. The universal such morphism, and thus the sought-for end, is then simply given by the identity wedge

$$\Big(\mathfrak{Map}(X,Y), \qquad \mathrm{id}\colon \mathfrak{Map}(X,Y) \dot{\to} \mathfrak{Map}(X,Y)\Big)$$

- For $\mathbf{M} = \mathbf{L} = \mathbf{R}$, we have

$$\mathfrak{Map}_{\mathbf{M}}(X,Y) = \mathrm{Map}(X,Y)$$

Indeed, we recall the adjunction

$$\mathrm{Hom}(S \odot X, Y) \cong \mathrm{sSet}(S, \mathrm{Map}(X,Y))$$

for all $S \in \mathrm{sSet}$ and $X, Y \in \mathrm{Psh}_\Delta(\mathbf{R})$. Thus, we only need to verify that $\mathfrak{Map}_{\mathbf{M}}(X,Y)$ gives rise to the same adjunction data:

$$\begin{aligned}
\mathrm{sSet}\Big(\Delta^n, \int_{m\in\mathbf{M}} \mathfrak{Map}(Xm, Ym)\Big) &\cong \int_{m\in\mathbf{M}} \mathrm{sSet}(\Delta^n, \mathrm{Map}(Xm, Ym)) \\
&\cong \int_{m\in\mathbf{M}} \mathrm{sSet}(\Delta^n \odot Xm, Ym) \\
&\cong \int_{m\in\mathbf{M}} \int_{l\in\Delta} \mathrm{Set}(\Delta^n_l \odot (Xm)_l, (Ym)_l) \\
&\cong \mathrm{Hom}(\Delta^n \odot X, Y)
\end{aligned}$$

Since any simplicial set $A \in \mathrm{sSet}$ is a colimit of representables, the above adjunction already yields the full adjunction.

7.10.1. *Cores.* Recall that for a subset $S \subset \{1,\dots,d\}$ we wrote $S^{\mathfrak{c}} := \{1,\dots,d\}\backslash S$ for its corresponding complement.

Definition 7.133. For $S \subset \{1,\dots,d\}$ and $\mathbf{m} \in \Delta^S$ a multisimplex, the functor

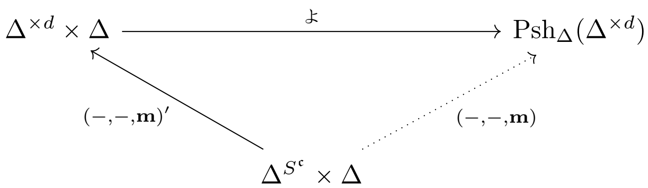
$$\mathrm{Psh}_\Delta(\Delta^{\times d}) \xrightarrow{\quad \mathfrak{ev}_{\mathbf{m}} \quad} \mathrm{Psh}_\Delta(\Delta^{S^{\mathfrak{c}}})$$

which takes a simplicial presheaf X on $\Delta^{\times d}$ to the partial evaluation $X\mathbf{m}$ at $\mathbf{m}$. The functor $\mathfrak{ev}_{\mathbf{m}}$ is called the *partial evaluation functor* for the multisimplex $\mathbf{m}$.

Recall the Nerve Realization paradigm 2.28. We then note that the partial evaluation functor $\mathfrak{ev}_{\mathbf{m}}$ arises as a corresponding nerve functor. Indeed, let us consider the functor

$$(-,-,\mathbf{m})'\colon \Delta^{S^{\mathfrak{c}}} \times \Delta \to \Delta^{\times d} \times \Delta$$

which takes a multisimplex $(\mathbf{a},[l]) \in \Delta^{S^{\mathfrak{c}}} \times \Delta$ and maps it onto the multisimplex $(\mathbf{a},[l],\mathbf{m})' \in \Delta^{\times d} \times \Delta \cong \Delta^{\{1,\dots,d\}} \times \Delta^{\{\tilde{1}\}}$ which is given by $(\mathbf{a},[l],\mathbf{m})'|_S \equiv \mathbf{m}$ and $(\mathbf{a},[l],\mathbf{m})'|_{S^{\mathfrak{c}}} \equiv \mathbf{a}$ and $(\mathbf{a},[l],\mathbf{m})'|_{\{\tilde{1}\}} \equiv [l]$. We then consider the functor $(-,-,\mathbf{m})$ given by the composition



Lemma 7.134. *The functor* $\mathfrak{ev}_{\mathbf{m}}$ *from the previous definition admits a left adjoint*

$$\mathrm{Psh}_{\Delta}(\Delta^{S^{\mathfrak{c}}}) \underset{\mathfrak{ev}_{\mathbf{m}}}{\overset{\mathfrak{L}_{\mathbf{m}}}{\rightleftarrows}} \mathrm{Psh}_{\Delta}(\Delta^{\times d}) \quad (\mathfrak{L}_{\mathbf{m}} \dashv \mathfrak{ev}_{\mathbf{m}})$$

which is given as the left Kan extension of $(-,-,\mathbf{m})\colon \Delta^{S^{\mathfrak{c}}} \times \Delta \to \mathrm{Psh}_{\Delta}(\Delta^{\times d})$ *along the Yoneda embedding:*

$$\begin{array}{ccc} \Delta^{S^{\mathfrak{c}}} \times \Delta & \xrightarrow{(-,-,\mathbf{m})} & \mathrm{Psh}_{\Delta}(\Delta^{\times d}) \\ \Big\downarrow よ & \overset{\mathfrak{L}_{m}}{\nearrow} & \\ \mathrm{Psh}_{\Delta}(\Delta^{S^{\mathfrak{c}}}) & & \end{array}$$

More explicitly, $\mathfrak{L}_{\mathbf{m}}$ *takes* $Y \in \mathrm{Psh}_{\Delta}(\Delta^{S^{\mathfrak{c}}})$ *and maps it onto*

$$j\mathbf{m} \odot Y\colon \Delta^{\times d} \to \mathrm{sSet}, \qquad \Delta^{\times d} \ni \mathbf{n} \mapsto \coprod_{\Delta^{S}(\mathbf{n}_S,\mathbf{m})} Y\mathbf{n}_{S^{\mathfrak{c}}}$$

where $\mathbf{n}_S \in \Delta^S$ *denotes the multisimplex obtained by throwing away all simplices that are not indexed by an element of* S *and analogously for* $\mathbf{n}_{S^{\mathfrak{c}}}$.

Proof. Let us start by showing that the explicit formula $j\mathbf{m}\odot(-)$ for $\mathfrak{L}_{\mathbf{m}}$ yields a left adjoint for $\mathfrak{ev}_{\mathbf{m}}$. If Y is representable, that is, $Y \cong よ(\mathbf{k},[l])$ for $(\mathbf{k},[l]) \in \Delta^{\times d} \times \Delta$, then

$$\mathrm{Hom}(\mathfrak{L}_{\mathbf{m}}Y, X) \cong X(\mathbf{k},\mathbf{m})_l \cong \mathrm{Hom}(Y, X(\mathbf{m}))$$

which already establishes the adjunction $\mathfrak{L}_{\mathbf{m}} \dashv \mathfrak{ev}_{\mathbf{m}}$. That $\mathfrak{L}_{\mathbf{m}}$ is the corresponding left Kan extension immediately follows from the fact that for $X \in \mathrm{Psh}_{\Delta}(\Delta^{\times d})$ we have

$$\mathfrak{N}_{(-,-,\mathbf{m})}X := \mathrm{Hom}((-,-,\mathbf{m}), X) \cong X(\mathbf{m}) = \mathfrak{ev}_{\mathbf{m}}(X)$$

and hence we are in the typical nerve-realization paradigm 2.28. □

We note that in particular if $\mathbf{m} = \mathbf{0} \in \Delta^S$ is the 0-multisimplex, then $\mathfrak{L}_{\mathbf{0}}$ is given by viewing a simplicial presheaf $Y \in \mathrm{Psh}_{\Delta}(\Delta^{S^{\mathfrak{c}}})$ as a simplicial presheaf on $\Delta^{\times d}$ by letting it be constant on all those factors in Δ^S. Hence any $S \subset \{1,\ldots,d\}$ gives rise to an adjunction

$$\mathrm{Psh}_{\Delta}(\Delta^{\times d} \times \mathit{\Gamma} \times \mathrm{Cart}) \underset{\mathfrak{ev}_S}{\overset{\mathfrak{L}_S}{\rightleftarrows}} \mathrm{Psh}_{\Delta}(\Delta^{S^{\mathfrak{c}}} \times \mathit{\Gamma} \times \mathrm{Cart})$$

where $\mathfrak{ev}_S$ denotes partial evaluation at $\mathbf{0} \in \Delta^S$. We then have the following:

Lemma 7.135 ([17] Lemma 2.2.8). *The adjunction* $\mathfrak{L}_S \dashv \mathfrak{ev}_S$ *from above descends to a Quillen adjunction at the level of the local injective model structures.*

The preceding Lemma justifies the following definition:

Definition 7.136. The functor

$$\mathrm{Psh}_{\Delta}(\Delta^{\times d} \times \mathit{\Gamma} \times \mathrm{Cart}) \xrightarrow{(-)^{\times}} \mathrm{Psh}_{\Delta}(\mathit{\Gamma} \times \mathrm{Cart})$$

is given as the right derived functor of the evaluation at $\mathbf{0}$-functor $\mathfrak{ev}_{\{1,\ldots,d\}}$. More concisely, $(-)^{\times} = \mathbb{R}\mathfrak{ev}_{\{1,\ldots,d\}}$:

$$\begin{array}{ccccc} \mathscr{C}^{\infty}\mathrm{Cat}^{\otimes,\mathrm{glob}}_{(\infty,d)} & \xrightarrow{\mathfrak{ev}_{\{1,\ldots,d\}}} & \mathscr{C}^{\infty}\mathrm{Grp}^{\otimes}_{\infty} & \twoheadrightarrow & \mathrm{Ho}(\mathscr{C}^{\infty}\mathrm{Grp}^{\otimes}_{\infty}) \\ & \searrow & \Downarrow & \nearrow_{\mathrm{loc}(\gamma_{\mathscr{D}}\mathbb{R}\mathfrak{ev}_{\{1,\ldots,d\}})=\mathbf{R}\mathfrak{ev}_{\{1,\ldots,d\}}} & \\ & & \mathrm{Ho}(\mathscr{C}^{\infty}\mathrm{Cat}^{\otimes,\mathrm{glob}}_{(\infty,d)}) & & \end{array}$$

Remark 7.137. Let $\mathscr{C}$ be a smooth symmetric monoidal (∞, d)-category, i.e., a fibrant object in $\mathscr{C}^{\infty}\mathrm{Cat}^{\otimes,\mathrm{glob}}_{(\infty,d)}$. Then the smooth symmetric monoidal ∞-groupoid $\mathscr{C}^{\times}$ is called the *underlying smooth symmetric monoidal ∞-groupoid of $\mathscr{C}$*. In particular, since $\mathscr{C}$ is fibrant and by using Ken Brown's Lemma 5.17, we deduce that $\mathscr{C}^{\times} \simeq \mathfrak{ev}_{\{1,\ldots,d\}}\mathscr{C} = \mathscr{C}(\mathbf{0})$.

8. Smooth Bordism Categories

> Do not meddle in the affairs of wizards, for they are subtle and quick to anger.
>
> J.R.R. Tolkien, The Fellowship of the Ring

The following Chapter is based on the corresponding construction of Bordism categories in [16].

In mathematics, the concept of bordism plays a crucial role in the study of manifolds and their embeddings. In recent years, the theory of bordisms has been extended to the setting of ∞-categories, giving rise to the theory of bordism ∞-categories. In this chapter, we will focus on a particular class of bordism ∞-categories, known as smooth ∞-bordism categories. Here, the term "smooth" refers to the smoothness of the ∞-categories involved, which are ∞-sheaves of ∞-categories. We will begin by introducing the general notion of a *geometric structure*, which will serve as a foundational concept for our study of smooth ∞-bordism categories as it will enable us to consider an onslaught of different flavors of bordism categories. We will then carry on with the construction of two variants of smooth ∞-bordism categories endowed with general geometric structures, as well as consider some low dimensional examples. Finally, we will study the behaviour and general properties of the mentioned bordim categories, as well as investigate their symmetric monoidal structure.

8.1. **Geometric Structures.**

Definition 8.1. Let FEmb_d be the category which has as objects submersions $p\colon M \twoheadrightarrow U$ with d-dimensional fibers (this means that $p^{-1}\{u\}$ is a d-dimensional manifold for all $u \in U$) and U an object in Cart. Morphisms are smooth bundle maps

$$(f\colon M \to N,\ F\colon U \to V)$$

that restrict to embeddings fiberwise, i.e., we have a commuting square

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ U & \xrightarrow[F]{} & V \end{array}$$

and the restriction $f_u\colon M_u \to N_{F(u)}$, where $M_u := p^{-1}\{u\}$ and $N_{F(u)} := q^{-1}\{F(u)\}$, is an open embedding. Moreover, this category may be looked at as a site by defining covering families to be those collections of morphisms

$$\begin{array}{ccc} M_i & \overset{\mathrm{i}_i}{\hookrightarrow} & M \\ \downarrow & & \downarrow \\ U_i & \underset{\mathrm{j}_i}{\hookrightarrow} & U \end{array}$$

such that the maps $\mathrm{i}_i, \mathrm{j}_i$ are open embeddings and the collection $\{\iota_i(M_i)\}_{i\in I}$ is an open cover of M.

Example 8.2. Consider the category Emb_d which has as objects smooth d-dimensional manifolds and morphisms are smooth embeddings. We have an embedding (a fully faithful functor)

$$\mathrm{Emb}_d \hookrightarrow \mathrm{FEmb}_d$$

$$\begin{array}{ccccc} M & & M & \overset{f}{\hookrightarrow} & N \\ {\scriptstyle f}\big\downarrow & \longmapsto & \big\downarrow\!\!\!\downarrow & & \big\downarrow\!\!\!\downarrow \\ N & & \mathbb{R}^0 & = & \mathbb{R}^0 \end{array}$$

which sends an object M to the canonical projection $M \twoheadrightarrow \mathbb{R}^0$, while an embedding is mapped to the bundle map $(f, \mathrm{id}_{\mathbb{R}^0})$.

Definition 8.3. A *fiberwise d-dimensional geometric structure* is a simplicial presheaf on FEmb_d.

Remark 8.4. A *d-dimensional topological structure* is an object in $\mathrm{Psh}_\Delta(\mathrm{Emb}_d)$.

Remark 8.5. There is a simplicial enrichment of the site FEmb_d: The *simplicially enriched site* $\mathfrak{FEmb}_d$ has the same objects as FEmb_d. Given two objects $M \twoheadrightarrow U$ and $N \twoheadrightarrow V$, the corresponding Hom-object

$$\mathfrak{FEmb}_d(M \twoheadrightarrow U, N \twoheadrightarrow V)$$

is the simplicial set whose n-simplices are pairs of smooth maps

$$(g\colon \delta^n \times M \to N,\ u\colon U \to V)$$

where $\delta^n := \{t \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1\}$, such that for any $t \in \delta^n$ the resulting map $g_t := g(t, -)\colon M \to N$ along with the map $u\colon U \to V$ form a morphism in FEmb_d:

$$\begin{array}{ccc} M & \xrightarrow{g_t} & N \\ \big\downarrow\!\!\!\downarrow & & \big\downarrow \\ U & \xrightarrow[u]{} & V \end{array}$$

In particular, a morphism $f\colon [n] \to [n']$ in Δ is mapped to the map

$$f^\star\colon \mathfrak{FEmb}_d(M \twoheadrightarrow U, N \twoheadrightarrow V)_{n'} \to \mathfrak{FEmb}_d(M \twoheadrightarrow U, N \twoheadrightarrow V)_n$$

$$(g\colon \delta^{n'} \times M \to N,\ u\colon U \to V) \mapsto (\delta^n \times M \xrightarrow{|f|\times \mathrm{id}_M} \delta^{n'} \times M \xrightarrow{g} N,\ u\colon U \to V)$$

Covering families for the enriched site $\mathfrak{FEmb}_d$ are the same as for Femb_d.

Definition 8.6. A *fiberwise d-dimensional geometric structure with isotopies* is a simplicial presheaf on the enriched site $\mathfrak{FEmb}_d$, that is, an sSet-enriched functor $\mathfrak{FEmb}_d^{\mathrm{op}} \to \mathrm{sSet}$.

Remark 8.7. There are two reasons for calling an object in the sSet-enriched functor category $\mathrm{Psh}_\Delta(\mathfrak{FEmb}_d)$ a geometric structure with isotopies. The main reason is that later we will see how any such object will induce a different variant of a smooth bordism (∞, d)-category which incorporates (higher) isotopies as higher morphisms. Another reason is that any such enriched functor $\mathbf{S}$ yields morphisms of simplicial sets

$$\mathfrak{FEmb}_d(M \twoheadrightarrow U, N \twoheadrightarrow V) \to \mathrm{Map}(\mathbf{S}(N \twoheadrightarrow V), \mathbf{S}(M \twoheadrightarrow U))$$

Recall that an l-simplex in the source is a pair $(g\colon \delta^l \times M \twoheadrightarrow N, u\colon U \to V)$. Here g is essentially a δ^l-family of embeddings, that is, an isotopy. Our functor $\mathbf{S}$ then takes (g, u) to a (higher) homotopy $\Delta^l \times \mathbf{S}(N \twoheadrightarrow V) \to \mathbf{S}(M \twoheadrightarrow U)$.

We will motivate how the above definition really incorporates a notion of *geometric structure* by considering examples:

Example 8.8. The *trivial geometric structure* is given by the terminal simplicial presheaf $\mathbf{S} = \star$.

Example 8.9. Let $\mathfrak{X}$ be a smooth manifold. Then $\mathfrak{X}$ may be interpreted as a geometric structure via the sheaf which assigns

$$(M \twoheadrightarrow U) \mapsto \mathscr{C}^\infty(M, \mathfrak{X})$$

to all submersions $M \twoheadrightarrow U$. A morphism

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ U & \xrightarrow[F]{} & V \end{array}$$

is mapped to the precomposition map

$$f^\star\colon \mathscr{C}^\infty(N, \mathfrak{X}) \to \mathscr{C}^\infty(M, \mathfrak{X})$$

Example 8.10. Recall that a *framing* for a d-manifold M is a *trivialization* of the tangent bundle of M. More concretely, consider the trivial vector bundle $\underline{\mathbb{R}}^d := M \times \mathbb{R}^d$ over M, then a framing is the data of an isomorphism $TM \cong \underline{\mathbb{R}}^d$:

$$\begin{array}{ccc} TM & \longrightarrow & \underline{\mathbb{R}}^d \\ & \searrow \; \cong \; \swarrow & \\ & M & \end{array}$$

More generally, if $d \leqslant d'$, then a d'*-framing* of a d-manifold M is a trivialization of the *stabilized tangent bundle* $TM \oplus \underline{\mathbb{R}}^{d'-d}$, that is, an isomorphism

$$\begin{array}{ccc} TM \oplus \underline{\mathbb{R}}^{d'-d} & \longrightarrow & \underline{\mathbb{R}}^{d'} \\ & \searrow \; \cong \; \swarrow & \\ & M & \end{array}$$

In order to encode the notion of a d-framing we consider the canonical projection map $(\mathbb{R}^d \times \mathbb{R}^0 \twoheadrightarrow \mathbb{R}^0) \in \mathrm{FEmb}_d$, apply the Yoneda embedding and view it as a simplicial presheaf on FEmb_d. Denote the resulting object by $j(\mathbb{R}^d \times \mathbb{R}^0 \twoheadrightarrow \mathbb{R}^0) \in \mathrm{Psh}_\Delta(\mathrm{FEmb}_d)$. We then note that any d-dimensional manifold may be viewed as an object in FEmb_d by looking at the canonical submersion with d-dimensional fibers $M \twoheadrightarrow \mathbb{R}^0$. Evaluating the presheaf $j(\mathbb{R}^d \twoheadrightarrow \mathbb{R}^0)$ at the manifold $M \twoheadrightarrow \mathbb{R}^0$ results in the set

$$\mathrm{FEmb}_d(M \twoheadrightarrow \mathbb{R}^0, \mathbb{R}^d \twoheadrightarrow \mathbb{R}^0)$$

which is precisely given by the set of embeddings $M \to \mathbb{R}^d$. Taking the tangent map of any such embedding $f\colon M \to \mathbb{R}^d$ results in linear isomorphisms $T_x f\colon T_x M \to T_x\mathbb{R}^d \cong \mathbb{R}^d$ and therefore these collect into a bundle isomorphism $TM \xrightarrow{Tf} \underline{\mathbb{R}}^d$, which is precisely the notion of a framing. For a d'-framing, with $d \leqslant d'$, we take $\widetilde{M} := M \times \mathbb{R}^{d'-d}$ and then consider the presheaf $j(\mathbb{R}^{d'} \twoheadrightarrow \mathbb{R}^0)$. Elements in the set $\mathrm{FEmb}_d(\widetilde{M} \twoheadrightarrow \mathbb{R}^0, \mathbb{R}^d \twoheadrightarrow \mathbb{R}^0)$ then give rise to d'-framings for M.

Example 8.11. More generally, if M is a d-dimensional manifold and U is a cartesian space, then we may consider the canonical submersion with d-dimensional fibers given by the projection map $M \times U \twoheadrightarrow U$. Evaluating the representable simplicial presheaf $j(\mathbb{R}^d \times U \twoheadrightarrow U)$ at $M \times U \twoheadrightarrow U$ then corresponds to the set of fiberwise embeddings

$$\begin{array}{ccc} M \times U & \xrightarrow{f} & \mathbb{R}^d \times U \\ \downarrow & & \downarrow \\ U & = & U \end{array}$$

Hence for each $u \in U$, we get an embedding $f_u \colon M \to \mathbb{R}^d$ and hence for each u we get a d-framing of M.

Example 8.12. Consider the presheaf

$$\mathrm{Riem}^{\mathfrak{f}}_d \colon \mathrm{FEmb}^{\mathrm{op}}_d \to \mathrm{Set}$$

of *fiberwise Riemannian metrics.* This presheaf sends an object $M \overset{p}{\twoheadrightarrow} U$ to the set of metrics on the *fiberwise tangent bundle*

$$T^{\mathfrak{f}(p)}M := \coprod_{u \in U} T(p^{-1}\{u\})$$

where $T(p^{-1}\{u\})$ denotes the usual tangent bundle of the d-dimensional manifold $p^{-1}\{u\}$. This means that an element in the set $\mathrm{Riem}^{\mathfrak{f}}_d(p)$ is a U-family

$$\{\mathfrak{m}^u\}_{u \in U}$$

of metrics $\mathfrak{m}^u$ on $T(p^{-1}\{u\})$. A morphism

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ U & \xrightarrow[F]{} & V \end{array}$$

in FEmb_d is sent to the function

$$(f, F)^\star \colon \mathrm{Riem}^{\mathfrak{f}}_d(N \overset{q}{\twoheadrightarrow} V) \to \mathrm{Riem}^{\mathfrak{f}}_d(M \overset{p}{\twoheadrightarrow} U)$$

which takes a V-family of metrics $\mathfrak{m} := \{\mathfrak{m}^v\}_{v \in V}$ on the fiberwise tangent bundle $T^{\mathfrak{f}(q)}N$ to the U-family of pullback metrics

$$(f, F)^\star \mathfrak{m} = \{f^\star \mathfrak{m}^{F(u)}\}_{u \in U}$$

where each member of this U-family is given by the formula

$$(f^\star \mathfrak{m}^{F(u)})_x(v, w) := \mathfrak{m}^{F(u)}_{f(x)}(T_x f(v), T_x f(w))$$

for all $x \in p^{-1}\{u\}$ and all $v, w \in T_x(p^{-1}\{u\})$, where $\mathfrak{m}^{F(u)}_{f(x)}$ denotes the symmetric bilinear form of the metric $\mathfrak{m}^{F(u)}$ at the point $f(x) \in N$. This is a well defined metric on the fiberwise tangent bundle $T^{\mathfrak{f}(p)}M$, since f is a fiberwise embedding. Analogously, one may define the presheaves

$$\mathrm{Lorentz}^{\mathfrak{f}}_d \colon \mathrm{FEmb}^{\mathrm{op}}_d \to \mathrm{Set}, \qquad \Psi\mathrm{Riem}^{\mathfrak{f}}_d \colon \mathrm{FEmb}^{\mathrm{op}}_d \to \mathrm{Set}$$

of *fiberwise Lorentzian manifolds* and more generally *fiberwise Pseudo-Riemannian metrics.*

Example 8.13. Let G be a Lie group. The notion of a *G-structure with connection* may be encoded as a simplicial presheaf $\mathbf{B}G_{\text{conn}}$ on FEmb_d by considering the *∞-sheafification* (a fibrant replacement) of the simplicial presheaf

$$(M \twoheadrightarrow U) \mapsto \mathfrak{N}(\Omega^1_U(M;\mathfrak{g}))//\mathscr{C}^\infty(M,G))$$

in $\text{Psh}_\Delta(\text{FEmb}_d)_{\check{\text{C}}\text{ech}}$. Here $\Omega^1_U(M;\mathfrak{g})$ denotes the set of *fiberwise Lie-algebra valued 1-forms*, $\mathscr{C}^\infty(M,G)$ denotes the group of smooth functions $M \to G$ and $\mathfrak{N}$ is the usual nerve functor which is applied to the *action groupoid*

$$\Omega^1_U(M;\mathfrak{g})//\mathscr{C}^\infty(M,G) \in \text{Cat}$$

which has

- objects given by the set of fiberwise smooth $\mathfrak{g}$-valued 1-forms $A \in \Omega^1_U(M;\mathfrak{g})$,
- morphisms $g\colon A \to A'$ are labeled by smooth functions $g \in \mathscr{C}^\infty(U,G)$ such that they relate source and target by a *gauge transformation*
$$A' = g^{-1}Ag + g^{-1}dg$$
where $g^{-1}Ag$ denotes the pointwise adjoint action of G on $\mathfrak{g}$ and $g^{-1}dg$ is the pullback $g^\star(\vartheta)$ of the *Maurer-Cartan form* $\vartheta \in \Omega^1(G;\mathfrak{g})$ (see the Nlab Maurer-Cartan form).
- Composition is induced by the group multiplication of G, i.e., composition of morphisms $g\colon A \to A'$ and $h\colon A' \to A''$ is given by the pointwise mutiplication $h \cdot g\colon A \to A''$.

For more details see [16] Example 3.3 and [13].

Example 8.14. We may encode *tangential structures* (see the Nlab tangential structure) in general as simplicial presheaves on FEmb_d. The corresponding construction may be found in [16] 3.2.

Example 8.15. Consider the enriched Yoneda embedding $よ\colon \mathfrak{FEmb}_d \hookrightarrow \text{Psh}_\Delta(\mathfrak{FEmb}_d)$. Then any object $(M \twoheadrightarrow U) \in \text{FEmb}_d$ induces a d-dimensional geometric structure with isotopies $よ(M \twoheadrightarrow U)$.

A good question to ask now is whether or not we can relate the categories $\text{Psh}_\Delta(\text{FEmb}_d)$ and $\text{Psh}_\Delta(\mathfrak{FEmb}_d)$ in some way. In particular, since FEmb_d and $\mathfrak{FEmb}_d$ are sites, we may consider the model categories $\text{Psh}_\Delta(\text{FEmb}_d)_{\check{\text{C}}\text{ech}}$ and $\text{Psh}_\Delta(\mathfrak{FEmb}_d)_{\check{\text{C}}\text{ech}}$ obtained by taking left Bousfield localizations of the corresponding injective model structures at the Čech nerves (see Definition 6.40) and ask whether there is a model categorical correspondence between these. To construct such a thing, consider the enriched functor

$$\begin{array}{ccc} \text{FEmb}_d & \xrightarrow{\ \rho\ } & \mathfrak{FEmb}_d \\ & & \\ (M \twoheadrightarrow U) & \longmapsto & (M \twoheadrightarrow U) \end{array}$$

(where Set is interpreted as an sSet-enriched category with Hom-objects being given by interpreting the usual Hom-sets as simplicial sets) which is the identity on objects, while we have a map of simplicial sets

$$\text{FEmb}_d(M \twoheadrightarrow U, N \twoheadrightarrow V) \to \mathfrak{FEmb}_d(M \twoheadrightarrow U, N \twoheadrightarrow V)$$

An l-simplex on the LHS, which is just a bundle map

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ U & \xrightarrow[u]{} & V \end{array}$$

is mapped to the δ^l-family

$$\begin{array}{ccc} M & \xrightarrow{g(t,-)} & N \\ \downarrow & & \downarrow \\ U & \xrightarrow[u]{} & V \end{array}$$

where $g\colon \delta^l \times M \to N$ is given by $g(t,-) := f$ for all $t \in \delta^l$. Taking the left Kan extension along ρ, denoted $\rho_!$ yields an enriched adjunction

$$\mathrm{Psh}_\Delta(\mathrm{FEmb}_d) \underset{\rho^\star}{\overset{\rho_!}{\rightleftarrows}} \mathrm{Psh}_\Delta(\mathfrak{FEmb}_d)$$

where $\rho^\star$ denotes precomposition with ρ (see Proposition 4.46). Explicitly, the simplicially enriched left adjoint functor $\rho_!$ sends a representable presheaf $j(M \twoheadrightarrow U)$ to the representable simplicially enriched presheaf よ$(M \twoheadrightarrow U)$. We then have the following:

Proposition 8.16. *The adjunction*

$$\mathrm{Psh}_\Delta(\mathrm{FEmb}_d)_{\check{\mathrm{C}}\mathrm{ech}} \underset{\rho^\star}{\overset{\rho_!}{\rightleftarrows}} \mathrm{Psh}_\Delta(\mathfrak{FEmb}_d)_{\check{\mathrm{C}}\mathrm{ech}}$$

is a Quillen adjunction.

Proof. This is Proposition 3.4.11 in the updated version of [16]. □

Remark 8.17. All aforementioned geometric structures may be considered as geometric structures with isotopies by applying $\rho_!$.

Example 8.18. There is a more direct construction to have a geometric structure of, say, Riemannian metrics with isotopies. Indeed, we define the simplicially enriched presheaf

$$\mathfrak{Riem}^{\mathfrak{f}}_d\colon \mathfrak{FEmb}_d^{\mathrm{op}} \to \mathrm{sSet}$$

An object $(M \overset{p}{\twoheadrightarrow} U) \in \mathfrak{FEmb}_d$ is sent to the simplicial set $\mathfrak{Riem}^{\mathfrak{f}}_d(p)$ which has as its set of l-simplices δ^l-families of fiberwise Riemannian metrics, that is, elements of the set

$$\prod_{t \in \delta^l} \mathrm{Riem}^{\mathfrak{f}}_d(p) \cong \mathrm{Set}(\delta^l, \mathrm{Riem}^{\mathfrak{f}}_d(p))$$

where $\delta^l := \{x \in \mathbb{R}^{l+1} \mid \sum_i x_i = 1\}$. In particular, the respective face and degeneracy maps d_k, s_k are induced by precomposition with the maps $|d^k|, |s^k|$ defined analogously as in equation (1):

$$\delta^{l-1} \xrightarrow{|d^k|} \delta^l \longrightarrow \mathrm{Riem}^{\mathfrak{f}}_d(p)$$

$$\delta^l \xrightarrow{|s^k|} \delta^{l+1} \longrightarrow \mathrm{Riem}^{\mathfrak{f}}_d(p)$$

The action on the simplicial set of morphisms

$$\mathfrak{FEmb}_d(M \overset{p}{\twoheadrightarrow} U, N \overset{q}{\twoheadrightarrow} V) \longrightarrow \mathrm{Map}(\mathfrak{Riem}_d^{\mathfrak{f}}(q), \mathfrak{Riem}_d^{\mathfrak{f}}(p))$$

has components

$$\mathfrak{FEmb}_d(M \overset{p}{\twoheadrightarrow} U, N \overset{q}{\twoheadrightarrow} V)_l \longrightarrow \mathrm{Map}(\mathfrak{Riem}_d^{\mathfrak{f}}(q), \mathfrak{Riem}_d^{\mathfrak{f}}(p))_l$$

which in turn send an element $(f_t, F)_{t\in\delta^l} \in \mathfrak{FEmb}_d(M \overset{p}{\twoheadrightarrow} U, N \overset{q}{\twoheadrightarrow} V)_l$ to the morphism $\Delta^l \times \mathfrak{Riem}_d^{\mathfrak{f}}(q) \to \mathfrak{Riem}_d^{\mathfrak{f}}(p)$ which, on n-simplices, is given by

$$\Delta^l_n \times \mathfrak{Riem}_d^{\mathfrak{f}}(q)_n \ni (h,\ \delta^l \overset{(\mathfrak{m}_t)_t}{\longrightarrow} \mathrm{Riem}_d^{\mathfrak{f}}(q)) \longmapsto \Big((f_{|h|(t)}, F)^\star \mathfrak{m}_t\Big)_{t\in\delta^n} \in \mathfrak{Riem}_d^{\mathfrak{f}}(p)_n$$

Analogously, we can define the simplicially enriched presheaves

$$\mathfrak{Lorentz}_d^{\mathfrak{f}} \colon \mathfrak{FEmb}_d^{\mathrm{op}} \to \mathrm{Set}, \qquad \Psi\mathfrak{Riem}_d^{\mathfrak{f}} \colon \mathfrak{FEmb}_d^{\mathrm{op}} \to \mathrm{Set}$$

of *fiberwise Lorentzian manifolds with isotopies* and more generally *fiberwise Pseudo-Riemannian metrics with isotopies.*

Example 8.19. We may also enrich Example 8.9. For a smooth manifold $\mathfrak{X}$ denote by $\mathfrak{C}^\infty(-, \mathfrak{X})$ the simplicially enriched geometric structure $\mathfrak{FEmb}_d^{\mathrm{op}} \to \mathrm{sSet}$ which takes a submersion $M \twoheadrightarrow U$ to the simplicial set

$$\mathfrak{C}^\infty(M, \mathfrak{X})$$

for which l-simplices are smooth δ^l-families of smooth maps $M \to \mathfrak{X}$, that is, a smooth map $\delta^l \times M \to \mathfrak{X}$. Face and degeneracy maps are given analogously to $\mathfrak{Cut}$. The action of this simplicially enriched functor on morphisms

$$\mathfrak{FEmb}_d(M \overset{p}{\twoheadrightarrow} U, N \overset{q}{\twoheadrightarrow} V) \longrightarrow \mathrm{Map}(\mathfrak{C}^\infty(N, \mathfrak{X}), \mathfrak{C}^\infty(M, \mathfrak{X}))$$

maps an l-simplex $(f_t, F)_{t\in\delta_l}$ on the LHS to an l-simplex $\Delta^l \times \mathfrak{C}^\infty(N, \mathfrak{X}) \to \mathfrak{C}^\infty(M, \mathfrak{X})$, which in turn has components

$$\begin{array}{ccc} \Delta^l_r \times \mathfrak{C}^\infty(N, \mathfrak{X})_r & \longrightarrow & \mathfrak{C}^\infty(M, \mathfrak{X})_r \\ (h, (\alpha_t)_{t\in\delta^r}) & \longmapsto & (f^\star_{|h|(t)}\alpha_t)_{t\in\delta^r} \end{array}$$

Remark 8.20. We do not necessarily need to use the model category of simplicial sets $\mathrm{sSet}_{\mathrm{Quillen}}$ as the codomain of our given geometric structures. In fact, a Quillen equivalent codomain would do just as nicely. One such choice is given by the *transferred model structure* on *smooth sets.* Indeed, consider the category of presheaves on $\mathrm{Set}^{\mathrm{Cart}^{\mathrm{op}}}$. This category of presheaves admits a model structure which is *transferred* from the Quillen model structure on sSet by means of the right adjoint

$$\begin{array}{ccc} \mathrm{Set}^{\mathrm{Cart}^{\mathrm{op}}} & \xrightarrow{\ \mathrm{Sing}_\infty\ } & \mathrm{sSet} \\ \mathfrak{F} & \longmapsto & \mathfrak{F}(\delta^\bullet) \cong \mathrm{Set}^{\mathrm{Cart}^{\mathrm{op}}}(よ \circ \delta^\bullet, \mathfrak{F}) \end{array}$$

referred to as the *smooth singular complex functor.* Here the functor $\delta^\bullet \colon \Delta \to \mathrm{Cart}$ is the map $[n] \mapsto \delta^n$ (which does the obvious thing to morphisms), while よ denotes the Yoneda embedding $\mathrm{Cart} \hookrightarrow \mathrm{Set}^{\mathrm{Cart}^{\mathrm{op}}}$. The resulting model category, called the model category of *smooth sets*, is denoted by $\mathscr{C}^\infty\mathrm{Set} = \mathrm{Set}^{\mathrm{Cart}^{\mathrm{op}}}_{\mathrm{transf}}$. Weak equivalences and fibrations in $\mathscr{C}^\infty\mathrm{Set}$ are those morphisms whose image under Sing_∞ are weak equivalences and fibrations in the Quillen model structure on simplicial sets. This model structure exists and it is cartesian. Moreover, by Theorem 2.28 Sing_∞ has a left adjoint $|-|_\infty$, and the corresponding adjunction is actually a Quillen equivalence

$$\mathscr{C}^\infty\mathrm{Set} \underset{\mathrm{Sing}_\infty}{\overset{|-|_\infty}{\underset{\longrightarrow}{\overset{\longleftarrow}{\scriptstyle \mathrm{Quillen}\perp}}}} \mathrm{sSet}_{\mathrm{Quillen}}$$

(this is Theorem 7.8 in [29]). Having all that, we may redefine the site $\mathfrak{FEmb}_d$ as something which is not enriched in simplicial sets, but rather enriched in smooth sets. The objects in $\mathfrak{FEmb}_d$ are the same as before. For objects $M \overset{p}{\twoheadrightarrow} U$ and $N \overset{q}{\twoheadrightarrow} V$, the corresponding Hom-smooth set is the smooth set which takes a cartesian space $L \in \mathrm{Cart}$ to the set of pairs of smooth maps $(f\colon L \times M \to N, F\colon U \to V)$ such that the resulting maps $(f(t,-), F)$ form morphisms in the usual category FEmb_d for all $t \in L$. A *d-dimensional geometric structure with isotopies* may then equivalently be defined as $\mathscr{C}^\infty\mathrm{Set}$-enriched presheaves $\mathfrak{FEmb}_d^{\mathrm{op}} \to \mathscr{C}^\infty\mathrm{Set}$. This definition will be employed in the newer versions of the papers [16, 17] as it is more convenient with regards to the given proofs in the papers.

We will now define very convenient subcategories of FEmb_d and $\mathfrak{FEmb}_d$, respectively.

Definition 8.21. Let $d \geqslant 0$.

- The site $\mathrm{FEmbCart}_d$ is the full subcategory of FEmb_d for which each object is isomorphic to some projection map $\mathbb{R}^d \times U \twoheadrightarrow U$, with the Grothendieck topology of good open covers (meaning open covers on total spaces such that any finite intersection is empty or isomorphic to an object of $\mathrm{FEmbCart}_d$).
- The simplicially enriched site $\mathfrak{FEmb}\mathrm{Cart}_d$ is the full enriched subcategory of $\mathfrak{FEmb}_d$ for which each objects is isomorphic to some projection $\mathbb{R}^d \times U$ with the Grothendieck topology of good open covers.
- The model categories $\mathrm{Psh}_\Delta(\mathrm{FEmbCart}_d)_{\check{\mathrm{C}}\mathrm{ech}}$ and $\mathrm{Psh}_\Delta(\mathfrak{FEmb}\mathrm{Cart}_d)_{\check{\mathrm{C}}\mathrm{ech}}$ are given as the respective left bousfield localizations of the injective model structrues at Čech covers.

We then have the following:

Proposition 8.22. *Denote by* $q\colon \mathrm{FEmbCart}_d \to \mathrm{FEmb}_d$ *and* $\mathfrak{q}\colon \mathfrak{FEmb}\mathrm{Cart}_d \to \mathfrak{FEmb}_d$ *the canonical inclusion functors. Then the induced restriction functors*

$$q^\star\colon \mathrm{Psh}_\Delta(\mathrm{FEmb}_d)_{\check{\mathrm{C}}\mathrm{ech}} \to \mathrm{Psh}_\Delta(\mathrm{FEmbCart}_d)_{\check{\mathrm{C}}\mathrm{ech}}$$
$$\mathfrak{q}^\star\colon \mathrm{Psh}_\Delta(\mathfrak{FEmb}_d)_{\check{\mathrm{C}}\mathrm{ech}} \to \mathrm{Psh}_\Delta(\mathfrak{FEmb}\mathrm{Cart}_d)_{\check{\mathrm{C}}\mathrm{ech}}$$

are right Quillen equivalences.

Proof. This is Proposition 3.3.2 in [16]. □

Remark 8.23. The previous proposition tells us that it is just as fine to define geometric structrues as simplicial presheaves on $\mathrm{FEmbCart}_d$ and $\mathfrak{FEmb}\mathrm{Cart}_d$, respectively.

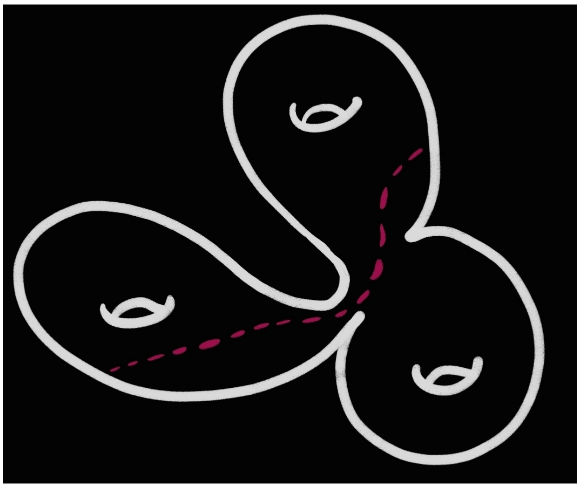
8.2. **The Smooth d-uple Bordism Category.** We start off this chapter by explaining the notion of a *cut* for a submersion $p\colon M \twoheadrightarrow U$. This will be generalized to *cut* $[m]$-tuples and cut $\mathbf{m}$-grids, for $\mathbf{m} \in \Delta^{\times d}$. A cut $\mathbf{m}$-grid for a d-dimensional manifold is precisely what one would expect: a grid of cuts of the given manifold which partitions the manifold into several pieces and in that sense makes the *core* of the cut-grid into a *manifold with corners.* This will then lead to a quite comfortable definition of bordism ∞-categories.

Definition 8.24. A *cut* of an object $M \overset{p}{\twoheadrightarrow} U$ in FEmb_d is a triple $(C_<, C_=, C_>)$ of subsets of M such that there is a smooth map $h\colon M \to \mathbb{R}$ satisfying

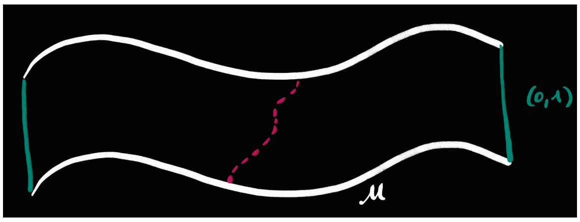
$$h^{-1}(-\infty, 0) = C_<, \qquad h^{-1}\{0\} = C_=, \qquad h^{-1}(0,\infty) = C_>$$

Moreover, we demand that the *fiberwise-regular values* of the map $(h,p)\colon M \to \mathbb{R} \times U$ form an open neighborhood of $\{0\} \times U$ in $\mathbb{R} \times U$. By fiberwise-regular values of (h,p) we mean the regular values of the maps $(h,p)|_{p^{-1}\{u\}}$ for all $u \in U$.

Example 8.25. A cut for the projection $M \twoheadrightarrow \mathbb{R}^0$ onto the 0-dimensional manifold $\mathbb{R}^0$ (a singleton) may look like



where M is the genus 3-surface as depicted. The red dashed line depicts the cut $C_=$. For $C_>$ and $C_<$ there is a choice to make depending on the function which induces the cut triple. $C_<$ could either be the half of the surface with two holes, or the half with only one hole and vice versa for $C_>$. If, on the other hand, M is a 1-dimensional manifold and $U := (0,1) \cong \mathbb{R}$ in Cart, then a cut for the projection $M \times U \twoheadrightarrow U$ might look something like



So we get an $(0,1)$-indexed family of cuts (points) for the manifold M and this family of cuts varies smoothly. The condition that the fiberwise regular values form an open neighborhood of $\{0\} \times U$ asserts that the given cut $C_=$ is a 1-dimensional submanifold of M.

Notation 8.26. For a cut triple as above we shall use the notation

$$C_\leqslant := C_< \cup C_=, \qquad C_\geqslant := C_> \cup C_=$$

This induces a partial order on the set of cuts with $C \leqslant C'$ if and only if $C_\leqslant \subset C'_\leqslant$.

Remark 8.27. The definition of a cut gives rise to a presheaf on FEmb_d in the following way: There is a functor $\mathrm{Cut}\colon \mathrm{FEmb}_d^{\mathrm{op}} \to \mathrm{Set}$ that maps an object $M \overset{p}{\twoheadrightarrow} U$

to its set of cuts, and a morphism

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ U & \xrightarrow[F]{} & V \end{array}$$

is mapped to the map

$$\mathrm{Cut}(f, F) \colon \mathrm{Cut}(q) \to \mathrm{Cut}(p)$$

which takes a cut for $N \overset{q}{\twoheadrightarrow} V$ and maps it onto the triple

$$(f^{-1}C_{<},\ f^{-1}C_{=},\ f^{-1}C_{>})$$

This is well defined (that is, it defines a cut for $M \overset{p}{\twoheadrightarrow} U$). Indeed, if $h \colon N \to \mathbb{R}$ witnesses $(C_{<}, C_{=}, C_{>})$ as a cut for $N \overset{q}{\twoheadrightarrow} V$, then we may consider $h \circ f \colon M \to \mathbb{R}$. Certainly,

$$(h \circ f)^{-1}(-\infty, 0) = f^{-1}C_{<}, \qquad (h \circ f)^{-1}\{0\} = f^{-1}C_{=}, \qquad (h \circ f)^{-1}(0, \infty) = f^{-1}C_{>}$$

Since f is a fiberwise embedding, we again have that the fiberwise regular values of $h \circ f$ form an open neighborhood of $\{0\} \times U \subset \mathbb{R} \times U$.

Definition 8.28. Let $d \geqslant 0$ and $[m] \in \Delta$. A *cut* $[m]$*-tuple* C for $(M \overset{p}{\twoheadrightarrow} U) \in \mathrm{FEmb}_d$ is a collection of cuts

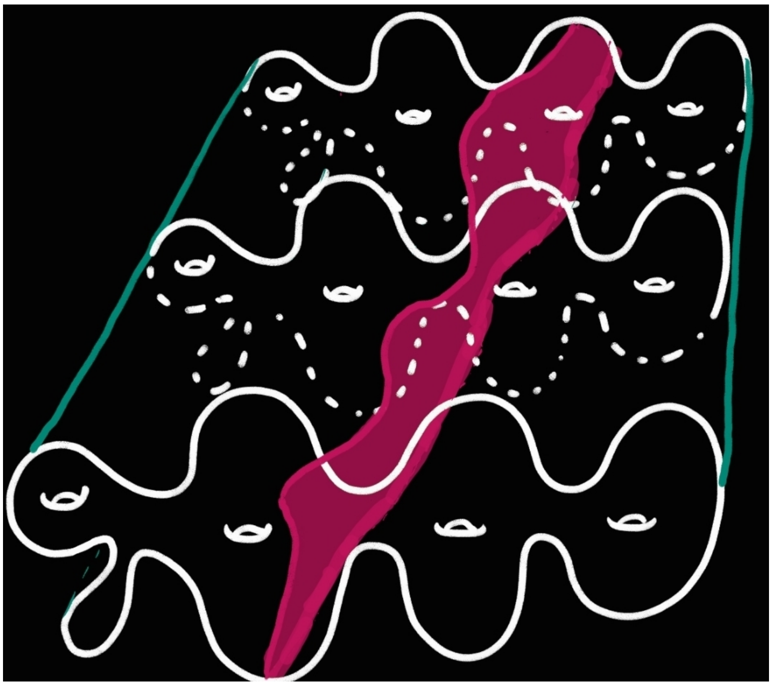
$$C_j = (C_{<j}, C_{=j}, C_{>j})$$

for $M \overset{p}{\twoheadrightarrow} U$ indexed by the vertices $j \in [m]$ such that

$$C_0 \leqslant C_1 \leqslant \ldots \leqslant C_m$$

Example 8.29. For $d = 2$, M a 4-genus surface and $[m] = [2]$, a cut $[m]$-tuple for $M \times \mathbb{R}^0 \twoheadrightarrow \mathbb{R}^0$ might look like

where the first dashed red line depicts $C_{=0}$, while the second one shows $C_{=1}$ and the third one shows $C_{=2}$. More generally, for $U = (0, 1) \cong \mathbb{R}$ in Cart a cut (in this case an area) for the projection $M \times U \twoheadrightarrow U$ might look like



Remark 8.30. Looking at the above definition more closely we realize:

- A cut $[m]$-tuple $C = (C_j)_{j \in [m]}$ also satisfies
$$C_{>m} \subset \ldots \subset C_{>0}$$
- We again obtain a functor Cut $\in \mathrm{Psh}_\Delta(\mathrm{FEmb}_d)$ that associates to an object $([m], M \overset{p}{\twoheadrightarrow} U)$ the set of cut $[m]$-tuples of p. To a morphism this functor associates a map of sets that takes preimages of the cuts and reindexes them according to the map of simplices. That is, a face map removes a cut and a degeneracy map duplicates a cut.

Notation 8.31. For a cut $[m]$-tuple as in the above definition, we write
$$C_{(j,j')} := C_{>j} \cap C_{<j'}, \qquad C_{[j,j']} := C_{\geqslant j} \cap C_{\leqslant j'}$$
for $j \leqslant j' \in [m]$.

Having this notion of cuts of objects in FEmb_d, we shall start with a precursor to the ∞-*bordism categories* we are trying to build:

Definition 8.32. Let $d \geqslant 0$ and fix $(\mathbf{m}, \langle l \rangle, U) \in \Delta^{\times d} \times \Gamma \times \mathrm{Cart}$. The category $\mathrm{B}(\mathbf{m}, \langle l \rangle, U)$ is given by the following data:

- An object of the category $\mathrm{B}(\mathbf{m}, \langle l \rangle, U)$ is a *bordism* given by
  - A d-dimensional smooth manifold M (possibly open).
  - A d-tuple $C = (C^i)_{i=1}^d$ of cut $[m_i]$-tuples C^i for the projection $M \times U \twoheadrightarrow U$.
  - A chosen map $P\colon M \times U \to \langle l \rangle$, which partitions the set of connected components of $M \times U$ into l disjoint subsets and another subset corresponding to the basepoint $\star$ (the slot $P^{-1}\{\star\}$ is referred to as the *trash bin*).

  Such an object must satisfy the *transversality property*:
  - For every subset $S \subset \{1, \ldots, d\}$ and for any map $j\colon S \to \mathbb{N}$ with $j_i \leqslant m_i$ for all $i \in S$, there is a smooth map $h_j\colon M \times U \to \mathbb{R}^S$ such

that for any $i \in S$ the map

$$\pi_i \circ h_j : M \times U \to \mathbb{R}$$

where $\pi_i : \mathbb{R}^S \to \mathbb{R}$ is the projection onto the i-th factor, yields the j_i-th cut $C^i_{j_i}$ in the cut tuple C^i. In particular, the fiberwise-regular values of $(h_j, p) : M \times U \to \mathbb{R}^S \times U$ form an open neighborhood of $\{0\} \times U \subset \mathbb{R}^S \times U$.

The collection of cut tuples $C = (C^i)$ is then called a *cut* $\mathbf{m}$*-tuple*. For notational convenience, let

$$C_{[j,j']} := \bigcap_{i \in S} C_{[j_i, j'_i]}, \qquad C_{(j,j')} := \bigcap_{i \in S} C_{(j_i, j'_i)}$$

for all $j, j' : S \to \mathbb{N}$ with $j_i \leqslant j'_i \leqslant m_i$ for all $i \in S$.

For j and j' as before, we also set

$$\operatorname{core}(M, C, P, j \leqslant j') := C_{[j,j']} \backslash P^{-1}\{\star\}$$

and we require this set to be *fiberwise compact* for all choices of j, j', that is, for each $u \in U$, the intersection

$$\operatorname{core}(M, C, P, j \leqslant j') \cap p^{-1}\{u\}$$

is compact for all choices j, j'. We will omit P in the notation if there is no danger for ambiguity. In the case where $S = \{1, \ldots, d\}$ and $j \equiv 0$ and $j'_i = m_i$ for all i, we set

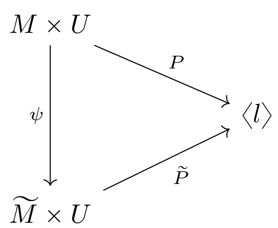
$$\operatorname{Core}(M, C, P) := C_{[j,j']} \backslash P^{-1}\{\star\}$$

and we call it the *core* of the bordism (M, C, P).

- A morphism in the category $\mathrm{B}(\mathbf{m}, \langle l \rangle, U)$ is a *cut respecting embedding*: That is, a morphism from a bordism (M, C, P) into a bordism $(\widetilde{M}, \widetilde{C}, \widetilde{P})$ is given by a smooth map $\psi : M \times U \to \widetilde{M} \times U$ *covering the identity on* U (that is $\psi = \psi_1 \times \mathrm{id}_U : M \times U \to \widetilde{M} \times U$) such that for all $u \in U$, the restriction $\psi : M \times \{u\} \to \widetilde{M} \times \{u\}$ is an embedding of smooth manifolds. Moreover, ψ satisfies the following properties:
  - For any $j, j' : \{1, \ldots, d\} \to \mathbb{N}$ with $j_i \leqslant j'_i \leqslant m_i$ for all i, there is an open set $Y_{j,j'} \subset M \times U$ containing the core $\operatorname{Core}(M, C, P, j \leqslant j')$ such that for any open subset $W_{j,j'} \subset Y_{j,j'}$ containing $\operatorname{Core}(M, C, P, j \leqslant j')$, the map ψ restricts to a fiberwise diffeomorphism

    $$W_{j,j'} \to \widetilde{W}_{j,j'}$$

    where $\widetilde{W}_{j,j'} \supset \operatorname{core}(\widetilde{M}, \widetilde{C}, \widetilde{P}, j \leqslant j')$ is open. Furthermore, we demand that the restriction of ψ to $W_{j,j'}$ satisfies $\psi(C^i_j) = \widetilde{C}^i_j$ for all cuts in the grid, meaning that after restricting the cut tuples C^i_j and $\widetilde{C}^i_j$ to $W_{j,j}$ and $\widetilde{W}_{j,j}$ respectively, the map ψ maps the subsets in the triple C^i_j to corresponding subsets in the triple $\widetilde{C}^i_j$.
  - ψ respects the partition maps P and $\widetilde{P}$ by ensuring commutativity of the diagram



Compostion of two such morphisms $\psi\colon M\times U\to\widetilde{M}\times U$ and $\varphi\colon\widetilde{M}\times U\to M'\times U$ is given by $(\varphi_1\circ\psi_1)\times\mathrm{id}_U\colon M\times U\to M'\times U$.

Remark 8.33. Some remarks are in order:

- The *transversality condition* precisely ensures that the cut $[m_i]$-tuples C^i intersect transversally with each other.
- The partition map $P\colon M\times U\to\langle l\rangle$ is allowed to have some slots being the empty set, that is, we allow $P^{-1}\{k\}=\varnothing$ for some $k\in\langle l\rangle$.
- The *trash bin* corresponds to those connected components of the manifold which are actually irrelevant for the bordism at hand.
- We may extend the above definition to obtain a functor

  $$\mathrm{B}\colon(\Delta^{\times d}\times\mathit{\Gamma}\times\mathrm{Cart})^{\mathrm{op}}\to\mathrm{Cat},\qquad(\mathbf{m},\langle l\rangle,U)\mapsto\mathrm{B}(\mathbf{m},\langle l\rangle,U)$$

  A coface map $d^k\colon[m_i-1]\to[m_i]$ in the i-th factor of $\Delta^{\times d}$ is mapped to the face map which removes the k-th cut from a given cut $[m_i]$-tuple (for the outer face maps this also shrinks the core appropriately). A codegeneracy map $s^k\colon[m_i+1]\to[m_i]$ in the i-th factor of the product $\Delta^{\times d}$ is sent to the degeneracy map that duplicates the k-th cut in a given cut $[m_i]$-tuple. For $\mathit{\Gamma}$, a map $\langle l\rangle\to\langle l'\rangle$ is simply composed with the given partition map $P\colon M\times U\to\langle l\rangle$ (this may possibly shrink the core). For a smooth map $\xi\colon V\to U$ (a morphism in Cart) we realize that the smooth map $\widetilde{\xi}:=\mathrm{id}_M\times\xi\colon M\times V\to M\times U$ defines a morphism

  $$(M\times V\twoheadrightarrow V)\xrightarrow{\widetilde{\xi}}(M\times U\twoheadrightarrow U)$$

  in FEmb_d. This morphism induces the functor

  $$\mathrm{B}(\mathbf{m},\langle l\rangle,\xi)\colon\mathrm{B}(\mathbf{m},\langle l\rangle,U)\to\mathrm{B}(\mathbf{m},\langle l\rangle,V)$$

  which takes an object (M,C,P) and maps it onto the bordism

  $$(M,\widetilde{\xi}^{-1}C,M\times V\xrightarrow{\widetilde{\xi}}M\times U\xrightarrow{P}\langle l\rangle)$$

  where

  $$\widetilde{\xi}^{-1}C:=(\widetilde{\xi}^{-1}C^i)_{i=1}^d,\qquad\widetilde{\xi}^{-1}C^i:=(\widetilde{\xi}^{-1}C^i_{<j_i},\widetilde{\xi}^{-1}C^i_{=j_i},\widetilde{\xi}^{-1}C^i_{>j_i})_{j_i\in[m_i]}$$

  which is a cut $\mathbf{m}$-tuple for $M\times V\twoheadrightarrow V$. A morphism $(M,C,P)\xrightarrow{\psi}(\widetilde{M},\widetilde{C},\widetilde{P})$ is mapped to

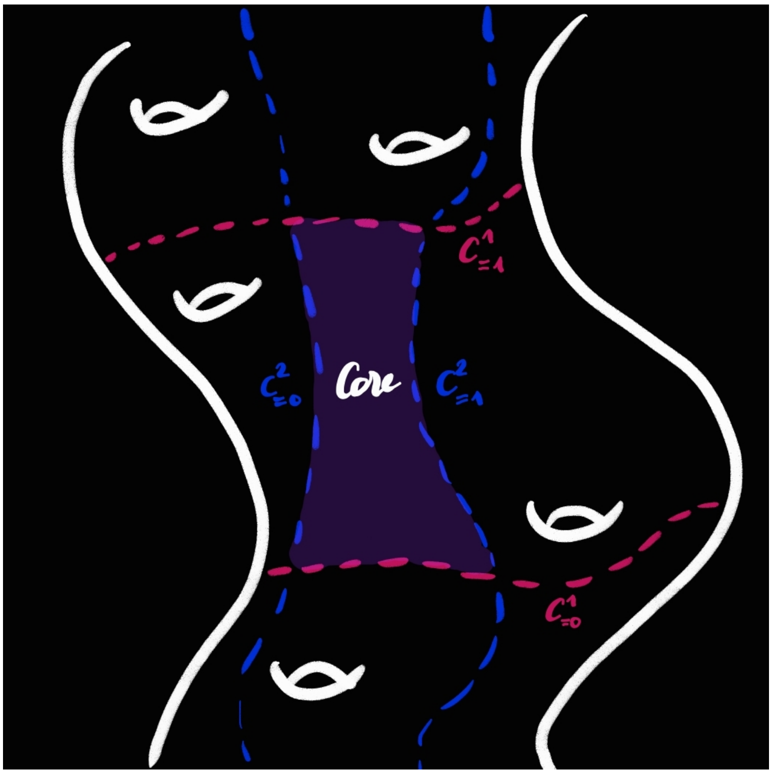
  $$(M,\widetilde{\xi}^{-1}C,P\circ\widetilde{\xi})\xrightarrow{\psi_1\times\mathrm{id}_V}(\widetilde{M},\widetilde{\xi}^{-1}\widetilde{C},\widetilde{P}\circ\widetilde{\xi})$$

Example 8.34. For $d=0$, an object in $B(\langle l\rangle,\mathbb{R}^0)$ is really just a 0-dimensional manifold (a disjoint union of points). Morphisms boil down to diffeomeorphisms of such 0-manifolds (bijections of finite sets).

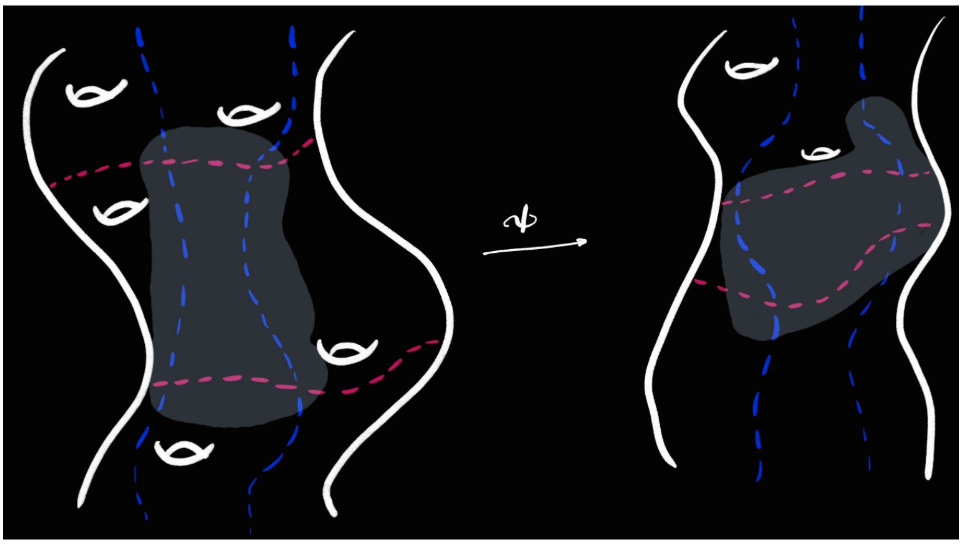
Example 8.35. Let $d=2$. Then an object in the category $\mathrm{B}(([1],[1]),\langle 1\rangle,\mathbb{R}^0)$ is given by a triple

$$(M,C=(C^1,C^2),P)$$

where M is a 2-dimensional manifold and C^1,C^2 are cut [1]-tuples for $M\cong M\times\mathbb{R}^0\twoheadrightarrow\mathbb{R}^0$. This could look like



where the partition map $P\colon M \to \langle 1 \rangle$ has preimages $P^{-1}\{1\} = M$ and $P^{-1}\{\star\} = \varnothing$. The core of this bordism is the area $(C^1_{>0} \cap C^1_{\leqslant 1}) \cap (C^2_{>0} \cap C^2_{\leqslant 1})$, i.e., the rectangular shape that is determined by the above cuts (the colored region). A morphism $\psi\colon M \to N$ between two such bordisms might look like



where ψ restricts to a diffeomorphism on the shadowed neighborhoods of the respective cores.

Definition 8.36. Let $d \geqslant 0$. The *d-uple bordism category* (with no geometric structure) is the object $\mathrm{Bord}_{(\infty,d),\mathrm{uple}}$ in the (model) category (see Definition 7.98)

$$\mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{uple}}_{(\infty,d)}$$

given by composition of the functor B with the nerve functor $\mathfrak{N}$:

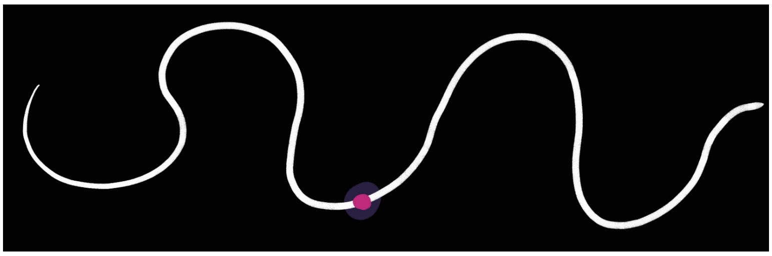
$$\begin{array}{ccc} (\Delta^{\times d} \times \Gamma \times \mathrm{Cart})^{\mathrm{op}} & \xrightarrow{\mathrm{B}} & \mathrm{Cat} \\ {\scriptstyle \mathrm{Bord}_{(d,\infty),\mathrm{uple}}} \downarrow & & \downarrow {\scriptstyle \mathfrak{N}} \\ \mathrm{sSet} & = & \mathrm{sSet} \end{array}$$

Remark 8.37. Note that we are not claiming that $\mathrm{Bord}_{(\infty,d),\mathrm{uple}}$ is a fibrant object in $\mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{uple}}_{(\infty,d)}$. For the time being, we shall pretend $\mathrm{Bord}_{(\infty,d),\mathrm{uple}}$ to be a smooth multiple symmetric monoidal (∞,d)-category. We will elaborate later why this is justified.

Example 8.38. Let $d = 1$. An object of $\mathrm{Bord}_{(\infty,1)}$ (in this case uple is redundant) is given by a vertex in $\mathrm{Bord}_{(\infty,1)}([0], \langle 1 \rangle, \mathbb{R}^0)$ (or more generally, replace $\mathbb{R}^0$ by $U \in \mathrm{Cart}$). Such a vertex is given by a triple

$$(M, C, P)$$

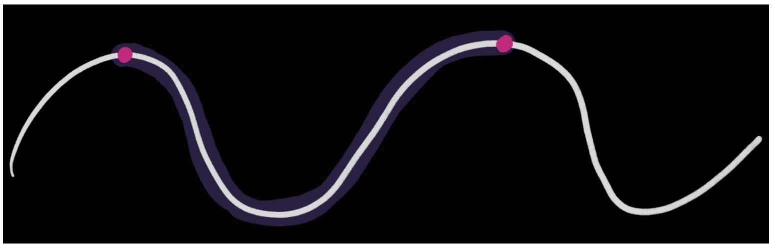
where M is a 1-dimensional manifold and C is a cut (a point) for $M \times \mathbb{R}^0 \twoheadrightarrow \mathbb{R}^0$. This could look like



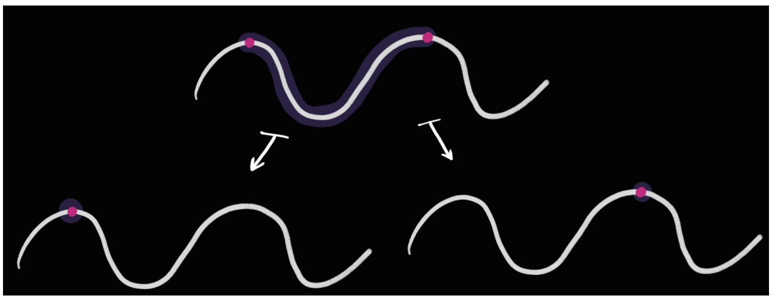
A 1-morphism in $\mathrm{Bord}_{(\infty,1),\mathrm{uple}}$ is given by a triple

$$(M, C = (C_1, C_2), P)$$

where M is a 1-dimensional manifold, and C_1, C_2 are cuts for the projection $M \times \mathbb{R}^0 \to \mathbb{R}^0$:



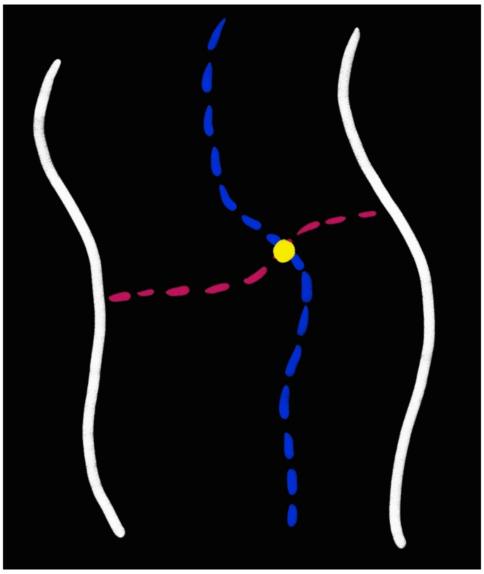
Domain and codomain maps are given by the maps $\mathrm{Bord}_{(\infty,d),\mathrm{uple}}(d^1, \langle 1 \rangle, \mathbb{R}^0)$ and $\mathrm{Bord}_{(\infty,1)}(d^0, \langle 1 \rangle, \mathbb{R}^0)$, which we can depict by



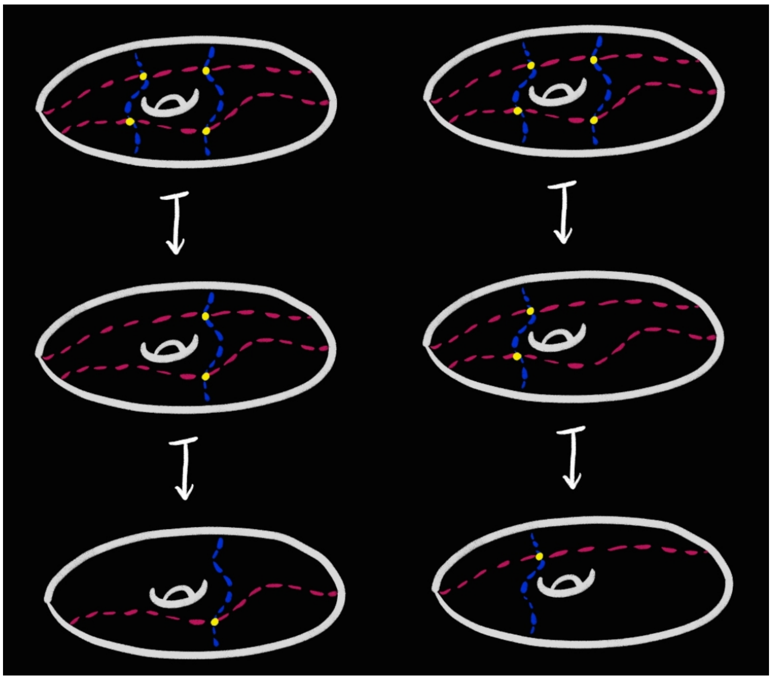
Example 8.39. Let $d = 2$. An object of $\mathrm{Bord}_{(\infty,2),\mathrm{uple}}$ is given by a vertex in $\mathrm{Bord}_{(\infty,2),\mathrm{uple}}(([0],[0]),\langle 1\rangle,\mathbb{R}^0)$ (or more generally, we could take $U \in \mathrm{Cart}$ arbitrary instead of $\mathbb{R}^0$). This in turn is given by a triple

$$(M, C = (C^1, C^2), P)$$

where M is a 2-dimensional manifold and C^1, C^2 are cuts for $M \cong M \times \mathbb{R}^0 \twoheadrightarrow \mathbb{R}^0$ and could look like

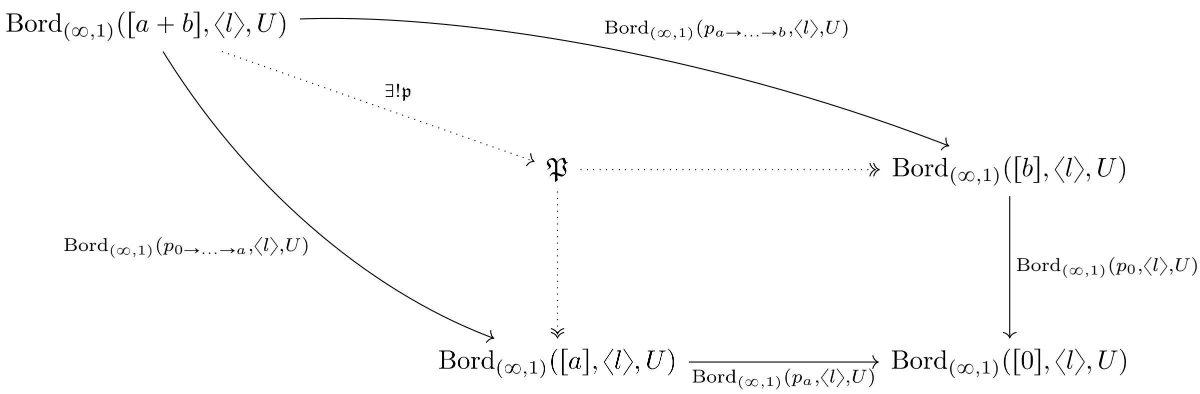


from which we realize that such an object simply boils down to a point of the manifold M (the core of the bordism) which is given by the intersection $C^1_= \cap C^2_=$. A general object (so if $U \in \mathrm{Cart}$ is arbitrary) is therefore a U-indexed smooth family of points in M. A 2-morphism in $\mathrm{Bord}_{(2,\infty),\mathrm{uple}}$ is given by a vertex of $\mathrm{Bord}_{(2,\infty),\mathrm{uple}}(([1],[1]),\langle 1\rangle,\mathbb{R}^0)$ (again we can make this more general by letting U be arbitrary) which might look like the top left (or right) torus with the pictured cuts:



The arrows pointing downwards to different images describe corresponding domain and codomain maps of horizontal morphisms (red lines) and vertical morphisms (blue lines). For example, looking at the left column, the first manipulation is applying the map $\mathrm{Bord}_{d,\mathrm{uple}}(([1],d^0),\langle 1\rangle,\mathbb{R}^0)$ to our given bordism. This says that the given 2-morphism has as its vertical codomain the 1-morphism as depicted in the middle left. The second manipulation is given by applying $\mathrm{Bord}_{(\infty,d),\mathrm{uple}}((d^1,[0]),\langle 1\rangle,\mathbb{R}^0)$. This yields the domain (a point of our manifold) of the given vertical 1-morphism. Analogously for the right column.

Remark 8.40. The simplicial presheaf $\mathrm{Bord}_{(\infty,d),\mathrm{uple}}$ satisfies Segal's special Δ-conditions: For notational convenience let us restrict to $d=1$. We then have an induced pullback diagram



where

$$\mathfrak{P} := \mathrm{Bord}_{(\infty,1)}([a],\langle l\rangle,U)\times_{\mathrm{Bord}_{(\infty,1)}([0],\langle l\rangle,U)}\mathrm{Bord}_{(\infty,1)}([b],\langle l\rangle,U)$$

is the corresponding pullback. The morphism $\mathfrak{p}$ (obtained by means of the universal property of $\mathfrak{P}$) is given by

$$\mathfrak{p} = (\mathfrak{p}_1,\mathfrak{p}_2)$$

where $\mathfrak{p}_1 := \mathrm{Bord}_{(\infty,1)}(p_{0\to\ldots\to a}, \langle l\rangle, U)$ and $\mathfrak{p}_2 := \mathrm{Bord}_{(\infty,1)}(p_{a\to\ldots\to b}, \langle l\rangle, U)$. Explicitly, the morphism $\mathfrak{p}$ takes vertices

$$(M, C = (C_0, \ldots, C_{a+b}), P)$$

in $\mathrm{Bord}_{(\infty,1)}([a+b], \langle l\rangle, U)$ to pairs of vertices

$$\Big[(M, (C_0, \ldots, C_a), P), (M, (C_a, \ldots, C_b), P)\Big] \in \mathfrak{P}$$

1-simplices in $\mathrm{Bord}_{(\infty,1)}([a+b], \langle 1\rangle, U)$ (cut-respecting embeddings)

$$(M, C, P) \xrightarrow{\psi} (\widetilde{M}, \widetilde{C}, \widetilde{P})$$

are mapped to 1-simplices in $\mathfrak{P}$ given by

$$\Big[(M, (C_0, \ldots, C_a), P) \xrightarrow{\psi} (\widetilde{M}, (\widetilde{C}_0, \ldots, C_a), P),\ (M, (C_a, \ldots, C_{a+b}), P) \xrightarrow{\psi} (\widetilde{M}, (\widetilde{C}_a, \ldots, \widetilde{C}_{a+b})), P)\Big]$$

For the remaining simplicial layers it is clear how $\mathfrak{p}$ acts (the higher layers are just composable chains of cut-respecting embeddings). A general vertex in the pullback $\mathfrak{P}$ is of the form

$$((M, C, P),\ (\widetilde{M}, \widetilde{C}, \widetilde{P})) \in \mathrm{Bord}_1([a], \langle l\rangle, U) \times \mathrm{Bord}_1([b], \langle l\rangle, U)$$

with the property that

$$\mathrm{Bord}_1(p_a, \langle l\rangle, U)((M, C, P)) = \mathrm{Bord}_1(p_0, \langle l\rangle, U)(\widetilde{M}, \widetilde{C}, \widetilde{P})$$

which is equivalent to

$$(M, C_a, P) = (\widetilde{M}, \widetilde{C}_0, \widetilde{P})$$

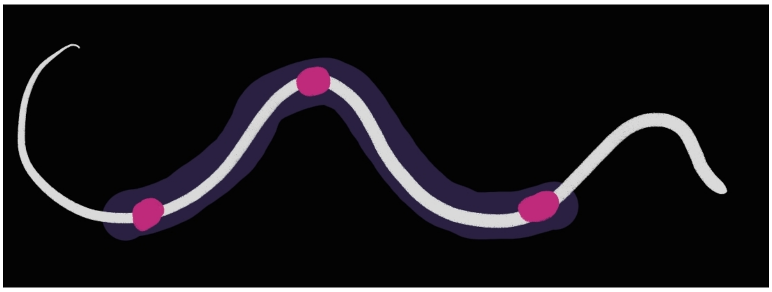
Hence $M = \widetilde{M}, P = \widetilde{P}$ and $C_a = \widetilde{C_0}$. In particular, the map $\mathfrak{p}$ has an obvious inverse: A vertex

$$((M, (C_0, \ldots, C_a), P), (M, (C_a, \ldots, C_{a+b}), P))$$

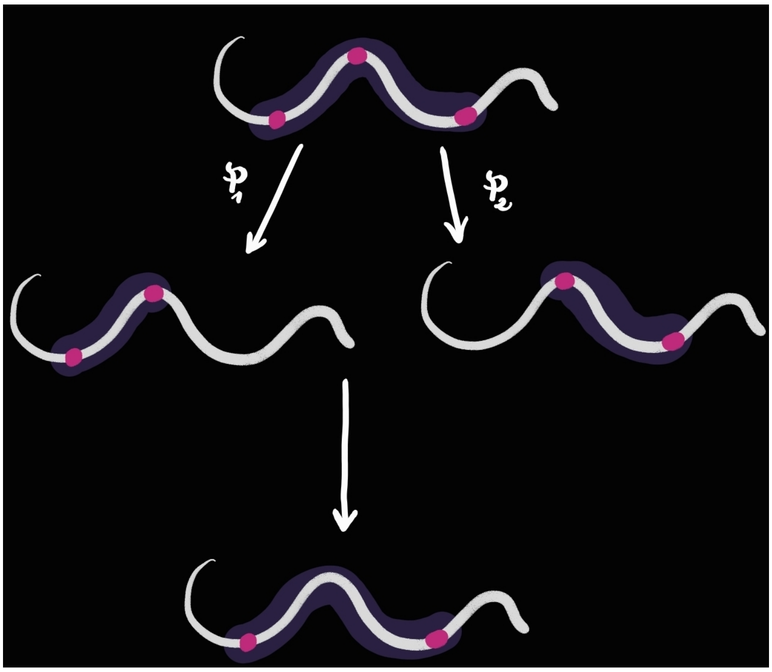
is mapped to

$$(M, (C_0, \ldots, C_{a+b}), P)$$

which verifies, in particular, that $\mathfrak{p}$ is a weak equivalence. This, however, is precisely the Segal condition. For $a = b = 1$ and $U = \mathbb{R}^0$, we get the following picture: A vertex in $\mathrm{Bord}_{(\infty,1)}([2], \langle 1\rangle, \mathbb{R}^0)$ is given by a picture like



Applying the map $\mathfrak{p}$ and then using the composition operation given by $\mathrm{Bord}_{(\infty,1)}(d^1, \langle 1\rangle, \mathbb{R}^0)$, for $d^1 \colon [1] \to [2]$, yields



So composition of 1-morphisms in $\mathrm{Bord}_{(\infty,1)}$ is just forgetting the middle cut.

8.3. **Bordism Categories with Geometric Structures.** In the previous Chapter we have defined bordisms with no additional structure. We would like to endow bordisms with geometric structure, that is, we want to imprint the structure of a simplicial presheaf $\mathbf{S} \in \mathrm{Psh}_\Delta(\mathrm{FEmb}_d)$ into the very fabric of the bordisms we consider. In order to define a corresponding object in the category $\mathrm{Psh}_\Delta(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})$ we again need a precursor:

Definition 8.41. Let $d \geqslant 0$ and let $\mathbf{S} \in \mathrm{Psh}_\Delta(\mathrm{FEmb}_d)$ be a geometric structure. For fixed $(\mathbf{m}, \langle l \rangle, U) \in \Delta^{\times d} \times \Gamma \times \mathrm{Cart}$, the simplicial object $\mathrm{B}^{\mathbf{S}}(\mathbf{m}, \langle l \rangle, U)$ in Cat is given by the following data:

- The simplicial set of objects is given by
$$\mathrm{Ob} := \coprod_{(M,C,P)} \mathbf{S}(M \times U \twoheadrightarrow U)$$
where the coproduct ranges over all objects (M, C, P) in Definition 8.32 with M a d-dimensional manifold, C a cut $\mathbf{m}$-tuple for the projection $M \times U \twoheadrightarrow U$ and P a partition $M \times U \to \langle l \rangle$ of connected components. In particular, (M, C, P) must satisfy the transversality condition.
- The simplicial set of morphisms is given by
$$\mathrm{Mor} := \coprod_{(M,C,P) \xrightarrow{\psi} (\widetilde{M}, \widetilde{C}, \widetilde{P})} \mathbf{S}(\widetilde{M} \times U \twoheadrightarrow U)$$
where the coproduct is taken over all the cut-respecting embeddings from Definition 8.32.
- The target map $\mathrm{cod}\colon \mathrm{Mor} \to \mathrm{Ob}$ sends the component indexed by a cut-respecting embedding $\psi\colon (M, C, P) \to (\widetilde{M}, \widetilde{C}, \widetilde{P})$ to itself by identity:
$$\mathrm{cod}_n\colon \mathrm{Mor}_n \to \mathrm{Ob}_n, \qquad (\psi,\ s \in \mathbf{S}(\widetilde{M} \times U \twoheadrightarrow U)_n) \mapsto ((\widetilde{M}, \widetilde{C}, \widetilde{P}), s)$$
Next, since
$$(\psi, \mathrm{id}_U)\colon (M \times U \twoheadrightarrow U) \to (\widetilde{M} \times U \twoheadrightarrow U)$$

constitutes a morphism in FEmb_d, the arrow

$$\mathbf{S}(\psi) := \mathbf{S}(\psi, \mathrm{id}_U) \colon \mathbf{S}(\widetilde{M} \times U \twoheadrightarrow U) \to \mathbf{S}(M \times U \twoheadrightarrow U)$$

makes sense. The source map $\mathrm{dom}\colon \mathrm{Mor} \to \mathrm{Ob}$ pulls back the component indexed by a cut-respecting embedding $\psi\colon (M, C, P) \to (\widetilde{M}, \widetilde{C}, \widetilde{P})$ by the morphism ψ via $\mathbf{S}$:

$$\mathrm{dom}_n \colon \mathrm{Mor}_n \to \mathrm{Ob}_n, \qquad (\psi,\ s \in \mathbf{S}(\widetilde{M} \times U \twoheadrightarrow U)_n) \mapsto ((M, C, P), \mathbf{S}(\psi)(s))$$

- Composition is induced by functoriality of $\mathbf{S}$: For morphisms

$$m_1 := ((M, C, P) \xrightarrow{\psi} (\widetilde{M}, \widetilde{C}, \widetilde{P}), s \in \mathbf{S}(\widetilde{M} \times U \twoheadrightarrow U)_n)$$
$$m_2 := ((\widetilde{M}, \widetilde{C}, \widetilde{P}) \xrightarrow{\varphi} (M', C', P'), s' \in \mathbf{S}(M' \times U \twoheadrightarrow U)_n)$$

in Mor_n such that $\mathrm{dom}_n(m_2) = \mathrm{cod}_n(m_1)$, we set

$$m_2 \circ m_1 := ((M, C, P) \xrightarrow{\varphi\circ\psi} (M', C', P'),\ s')$$

This is well defined, since

$$\begin{aligned}\mathrm{dom}_n(m_2 \circ m_1) &= ((M, C, P), \mathbf{S}(\varphi \circ \psi)(s')) \\ &= ((M, C, P), \mathbf{S}(\psi)S(\varphi)(s')) \\ &= ((M, C, P), \mathbf{S}(\psi)(s)) \\ &= \mathrm{dom}_n(m_1)\end{aligned}$$

and $\mathrm{cod}_n(m_2 \circ m_1) = ((M', C', P'), s') = \mathrm{cod}_n(m_2)$.

Remark 8.42. Let us point out some subtleties:

- If $\mathbf{S} = \star$ is the terminal functor (so we have no geometric structure), then $\mathrm{B}^{\mathbf{S}}(\mathbf{m}, \langle l\rangle, U)$ is just $\mathrm{B}(\mathbf{m}, \langle l\rangle, U)$ but interpreted as a constant simplicial object in Cat (recall Definition 8.32).
- We may extend the above definition to obtain a functor

$$\mathrm{B}^{\mathbf{S}} \colon (\Delta^{\times d} \times \mathit{\Gamma} \times \mathrm{Cart})^{\mathrm{op}} \to \mathrm{Cat}^{\Delta^{\mathrm{op}}}, \qquad (\mathbf{m}, \langle l\rangle, U) \mapsto \mathrm{B}^{\mathbf{S}}(\mathbf{m}, \langle l\rangle, U)$$

The coface map $d^k\colon [m_i - 1] \to [m_i]$ in the i-th factor of the product $\Delta^{\times d}$ removes the k-th cut in the corresponding cut $[m_i]$-tuple in the indexing triple (M, C, P) of the coproduct. Analogously, the codegeneracy map $s^k\colon [m_i] \to [m_i + 1]$ duplicates the k-th cut in the corresponding cut $[m_i]$-tuple in the indexing triple (M, C, P) of the coproduct. For $\mathit{\Gamma}$, a map $\langle l\rangle \to \langle l'\rangle$ is simply composed with the corresponding partition map in an indexing triple (M, C, P) in the coproduct. For a smooth map $\xi\colon V \to U$ in Cart an object in the n-th layer $((M, C, P),\ s \in \mathbf{S}(M \times U \twoheadrightarrow U)_n)$ is taken to

$$((M, \widetilde{\xi}^{-1}C, P \circ \widetilde{\xi}),\ \mathbf{S}(\widetilde{\xi})(f) \in \mathbf{S}(M \times V \twoheadrightarrow V)_n)$$

where we recall that $\widetilde{\xi} := \mathrm{id}_M \times \xi$. A morphism

$$((M, C, P) \xrightarrow{\psi} (\widetilde{M}, \widetilde{C}, \widetilde{P}), s \in \mathbf{S}(\widetilde{M} \times U \twoheadrightarrow U)_n)$$

in the n-th layer is taken to

$$((M, \widetilde{\xi}^{-1}C, P \circ \widetilde{\xi}) \xrightarrow{\psi_1\times\mathrm{id}_V} (\widetilde{M}, \widetilde{\xi}^{-1}\widetilde{C}, \widetilde{P} \circ \widetilde{\xi}),\ \mathbf{S}(\widetilde{\xi})(s) \in \mathbf{S}(\widetilde{M} \times V \twoheadrightarrow V)_n)$$

Definition 8.43. Fix $d \geqslant 0$ and let $\mathbf{S} \in \mathrm{Psh}_{\Delta}(\mathrm{FEmb}_d)$ be a geometric structure. The *d-uple bordism category with geometric structure* $\mathbf{S}$ is the object $\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$ in the (model) category (see Definition 7.98)

$$\mathscr{C}^{\infty}\mathrm{Cat}^{\otimes,\mathrm{uple}}_{(\infty,d)}$$

given by the following composition of functors:

$$\begin{array}{ccc} (\Delta^{\times d} \times \mathit{\Gamma} \times \mathrm{Cart})^{\mathrm{op}} & \xrightarrow{\mathrm{B}^{\mathbf{S}}} & \mathrm{Cat}^{\Delta^{\mathrm{op}}} \\ {\scriptstyle \mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}} \downarrow & & \downarrow {\scriptstyle \mathfrak{N}} \\ \mathrm{sSet} & \xleftarrow[\mathrm{diag}]{} & \mathrm{Psh}_{\Delta}(\Delta) \end{array}$$

where diag: $\mathrm{Psh}_{\Delta}(\Delta) \to \mathrm{sSet}$ takes the diagonal of a bisimplicial set

$$\mathrm{Psh}_{\Delta}(\Delta) \ni X \mapsto (\mathrm{diag}(X) \in \mathrm{sSet}, [n] \mapsto X_{n,n})$$

and $\mathfrak{N}\colon \mathrm{Cat}^{\Delta^{\mathrm{op}}} \to \mathrm{Psh}_{\Delta}(\Delta)$ takes the (usual) nerve levelwise.

Remark 8.44. If we have no geometric structure, that is, if $\mathbf{S} = \star$ is the terminal simplicial presheaf on FEmb_d, then $\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}} = \mathrm{Bord}_{(\infty,d),\mathrm{uple}}$.

Example 8.45. Consider $\mathrm{Bord}^{\mathrm{Riem}^{\mathfrak{f}}_1}_{(\infty,1)}$ (recall Example 8.12). Vertices in $\mathrm{Bord}^{\mathrm{Riem}^{\mathfrak{f}}_1}_{(\infty,1)}([1], \langle 1\rangle, U)$ are given by

$$\begin{aligned} \mathrm{Bord}^{\mathrm{Riem}^{\mathfrak{f}}_1}_{(\infty,1)}([1], \langle 1\rangle, U)_0 &= \mathrm{diag} \circ \mathfrak{N}(\mathrm{B}^{\mathrm{Riem}^{\mathfrak{f}}_1}([1], \langle 1\rangle, U))([0]) \\ &= \mathfrak{N}(\mathrm{B}^{\mathrm{Riem}^{\mathfrak{f}}_1}([1], \langle 1\rangle, U)_0)_0 \\ &= \mathrm{Ob}(\mathrm{B}^{\mathrm{Riem}^{\mathfrak{f}}_1}([1], \langle 1\rangle, U)_0) \\ &= \coprod_{(M,C,P)} \mathrm{Riem}^{\mathfrak{f}}_1(M \times U \twoheadrightarrow U) \end{aligned}$$

For $U = \mathbb{R}^0$, an element of $\mathrm{Riem}^{\mathfrak{f}}_1(M \times \mathbb{R}^0 \twoheadrightarrow \mathbb{R}^0)$ is precisely a Riemannian metric $\mathfrak{m}$ on M, and therefore a 1-morphism in $\mathrm{Bord}^{\mathrm{Riem}^{\mathfrak{f}}_1}_{(\infty,1)}$ (where we disregard the U-parameter by letting it be $\mathbb{R}^0$) is given by a quadruple

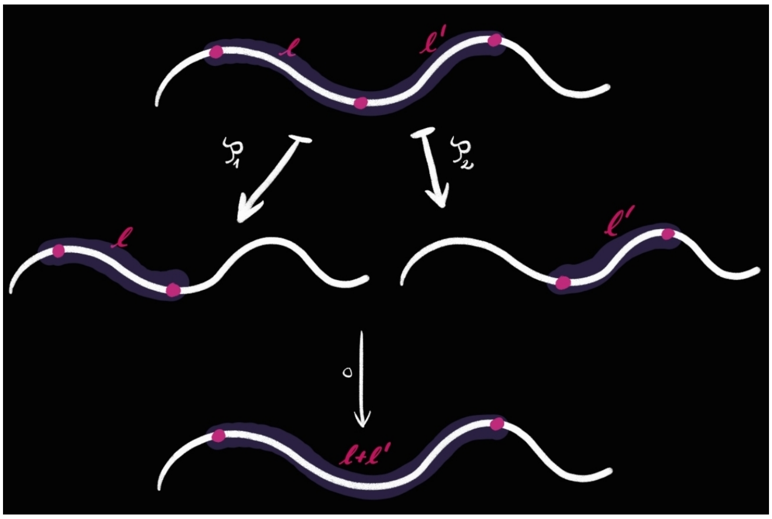
$$(M, C, P, \mathfrak{m})$$

The Riemannian metric $\mathfrak{m}$ then gives rise to an ordinary metric $d_{\mathfrak{m}}\colon M \times M \to \mathbb{R}_{\geqslant 0}$ on M given by

$$d_{\mathfrak{m}}(a, b) := \inf_{a \overset{\gamma}{\rightsquigarrow} b} \int_0^1 \sqrt{\mathfrak{m}(\dot\gamma, \dot\gamma)}\, dt$$

where γ is a path $[0, 1] \to M$ with $\gamma(0) = a$ and $\gamma(1) = b$. In particular, we have bordisms with lengths. For example, a 1-morphism may be depicted by

where l denotes the length assigned to the core of the depicted bordism. Composition of such 1-morphisms is then depicted by



which boils down to forgetting the middle cut and adding Riemannian lengths.

We note that the construction of $\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$ seems a bit weird at first glance. One reason for this is for example that for any triple $(\mathbf{m}, \langle l\rangle, U)$ the partial evaluation $\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}(\mathbf{m}, \langle l\rangle, U)$ is not a Kan complex, which is one of the crucial properties we have for a prospective (∞, d)-category. The reason for this is that we construct $\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$ by means of the nerve of a category. We can amend that however by passing to germs.

Definition 8.46. Let $\mathbf{S} \in \mathrm{Psh}_\Delta(\mathrm{FEmb}_d)$ be a d-dimensional geometric structure and consider a bordism (M, C, P), as in Definition 8.32, along with the canonical projection $p\colon M \times U \twoheadrightarrow U$.

- The $\mathbf{S}$-*germ associated to* (M, C, P) is given by
$$\mathbf{S}_C(M \times U \twoheadrightarrow U) := \operatorname*{colim}_{V \supset \mathrm{core}(M,C,P)} \mathbf{S}(V \twoheadrightarrow p(V))$$
where the colimit is taken over the poset of open subsets $V \subset M \times U$ containing the subset $\mathrm{core}(M, C, P)$.
- By functoriality of $\mathbf{S}$ we can pull back along fiberwise cut-respecting embeddings $\psi\colon M \times U \to \widetilde{M} \times U$, where we restrict to neighborhoods of the core. Indeed, by considering the diagrams
$$\begin{array}{ccc} \mathbf{S}(V) & \hookrightarrow & \mathbf{S}_C(M \times U \twoheadrightarrow U) \\ \uparrow{\scriptstyle \mathbf{S}(\psi)} & & \uparrow{\scriptstyle \mathbf{S}_C(\psi)} \\ \mathbf{S}(\widetilde{V}) & \hookrightarrow & S_C(\widetilde{M} \times U \twoheadrightarrow U) \end{array}$$
for $V \subset M \times U$ and $\widetilde{V}$ where ψ is interpreted to be its restriction to V, we obtain a map $\mathbf{S}_C(\psi)$.
- The simplicial object $\mathfrak{g}\mathrm{B}(\mathbf{m}, \langle l\rangle, U)$ in Grpd (the category of groupoids), for $(\mathbf{m}, \langle l\rangle, U) \in \Delta^{\times d} \times \mathit{\Gamma} \times \mathrm{Cart}$ is given by the following data:
  - The simplicial set of objects is given by
$$\mathrm{Ob} := \coprod_{(M,C,P)} \mathbf{S}_C(M \times U \twoheadrightarrow U)$$
where the coproduct ranges over the objects in Definition 8.32.

– The simplicial set of morphisms is given by

$$\mathrm{Mor} := \coprod_{\mathfrak{g}(\psi)} \mathbf{S}_C(\widetilde{M} \times U \twoheadrightarrow U)$$

where the coproduct is taken over all germs $\mathfrak{g}(\psi)$ of fiberwise cut-respecting embeddings $\psi\colon M \times U \twoheadrightarrow \widetilde{M} \times U$ from Definition 8.32. More in detail, two fiberwise cut respecting embeddings $\psi, \psi'\colon M \times U \twoheadrightarrow \widetilde{M} \times U$ are identified if they agree on an open neighborhood $V \subset M \times U$, which contains the subset $\mathrm{core}(M, C, P)$. The corresponding equivalence class is then denoted by $\mathfrak{g}(\psi)$. Target and domain maps for the individual simplicial layers, and the respective composition operations are given analogously as in Definition 8.41.

– The above assignment collects into a functor

$$\mathfrak{g}\mathrm{B}\colon (\Delta^{\times d} \times \Gamma \times \mathrm{Cart})^{\mathrm{op}} \to \mathrm{Grpd}, \qquad (\mathbf{m}, \langle l \rangle, U) \mapsto \mathfrak{g}\mathrm{B}(\mathbf{m}, \langle l \rangle, U)$$

– The *germy smooth (∞, d)-bordism category* $\mathfrak{g}\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$ is given by the composition of functors

$$\begin{array}{ccc} (\Delta^{\times d} \times \Gamma \times \mathrm{Cart})^{\mathrm{op}} & \xrightarrow{\mathfrak{g}\mathrm{B}^{\mathbf{S}}} & \mathrm{Grpd}^{\Delta^{\mathrm{op}}} \\ \Big\downarrow{\scriptstyle \mathfrak{g}\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}} & & \Big\downarrow{\scriptstyle \mathfrak{N}} \\ \mathrm{sSet} & \xleftarrow[\mathrm{diag}]{} & \mathrm{Psh}_{\Delta}(\Delta) \end{array}$$

similar to Definition 8.43.

It turns out that $\mathfrak{g}\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$ and $\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$ are equivalent in the proper sense. Indeed, consider the *germification map*

$$\mathfrak{germ}\colon \mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}} \to \mathfrak{g}\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$$

which sends a bordism to the $\mathbf{S}$-germ of its core. More precisely, for $(\mathbf{m}, \langle l \rangle, U) \in \Delta^{\times d} \times \Gamma \times \mathrm{Cart}$, the map of simplicial sets $\mathfrak{germ}(\mathbf{m}, \langle l \rangle, U)$ is given by applying $\mathrm{diag} \circ \mathfrak{N}$ to the functor

$$\mathrm{B}^{\mathbf{S}}(\mathbf{m}, \langle l \rangle, U) \to \mathfrak{g}\mathrm{B}^{\mathbf{S}}(\mathbf{m}, \langle l \rangle, U)$$

which maps objects (in some simplicial layer)

$$\Big((M, C, P),\ s \in \mathbf{S}(M \times U \twoheadrightarrow U)\Big) \mapsto \Big((M, C, P),\ \mathfrak{g}(s) \in \mathbf{S}_C(M \times U \twoheadrightarrow U)\Big)$$

where $\mathfrak{g}(s)$ is given by the image of s under the canonical map $S(M \times U \twoheadrightarrow U) \to S_C(M \times U \twoheadrightarrow U)$. For morphisms we have the assignment

$$((M, C, P) \xrightarrow{\psi} (\widetilde{M}, \widetilde{C}, \widetilde{P}),\ s \in \mathbf{S}(\widetilde{M} \times U \twoheadrightarrow U)) \mapsto (\mathfrak{g}(\psi), \mathfrak{g}(s))$$

sending both ψ and the geometric structure s to its corresponding equivalence classes. With these definitions we have the following result:

Proposition 8.47. *The germification map*

$$\mathfrak{germ}\colon \mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}} \to \mathfrak{g}\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$$

defines an objectwise weak equivalence, that is, the maps

$$\mathfrak{germ}(\mathbf{m}, \langle l \rangle, U)\colon \mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}(\mathbf{m}, \langle l \rangle, U) \to \mathfrak{g}\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}(\mathbf{m}, \langle l \rangle, U)$$

are weak equivalences of simplicial sets.

Proof. This is Proposition 4.2.4 in the updated version of [16]. □

Remark 8.48. The previous proposition verifies that both variants of bordism categories are equivalent, so that we can choose each of these for practical applications. However, it is very much clear that not having to care for germs all the time is much more comfortable.

Remark 8.49. The previous Proposition also makes clear that the crucial information of a bordism in $\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$ is fully contained in the core of the bordism and a germ of the geometric structure $\mathbf{S}$ around the core. For (M, C, P) a vertex in $\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}(\mathbf{m}, \langle l\rangle, U)$, we will call $M \times U$ the *ambient manifold* of our bordism, while $\mathrm{core}(M, C, P)$ is the information we really need. It seems as if the ambient manifold is entirely redundant here. This is not the case, as it provides quite a useful way to think about composition of bordisms with geometric structure, as we will explore later.

8.4. **Bordisms with Isotopies.** The reader familiar with the paper [24] might remember that higher morphisms in the corresponding bordism category defined there were isotopies (higher isotopies) between diffeomorphisms. We have another variant of the bordism category which incorporates isotopies between cuts and is therefore somewhat more reminiscient to [24]. This variant turns out to be the more important one. This requires further preliminary definitions and another precursor.

Definition 8.50. Let $l \in \mathbb{N}$.

- The *extended l-simplex* is the set
$$\delta^l := \{t \in \mathbb{R}^{l+1} \mid \sum_i t_i = 1\}$$
The compact part of δ^l will be denoted by δ^l_c and it is given by
$$\delta^l_c := |\Delta^l| = \{t \in \mathbb{R}^{l+1}_{\geqslant 0} \mid \sum_i t_i = 1\}$$
- A *δ^l-family of cuts* of an object $M \overset{p}{\twoheadrightarrow} U$ in FEmb_d is a collection of cuts
$$C := \{(C_<, C_=, C_>)_t \mid t \in \delta^l\}$$
with the property that there exists a smooth map $h\colon \delta^l \times M \to \mathbb{R}$ such that for all $t \in \delta^l$, the map $h_t := h(t, -)\colon M \to \mathbb{R}$ gives rise to the cut $(C_<, C_=, C_>)_t$ as in Definition 8.24. We identify two such δ^l-indexed collections if they have the same germ around the compact part of δ^l_c, i.e., if C and $\widetilde{C}$ are two δ^l-families of cuts, then $C \sim \widetilde{C}$ if there exists an open neighborhood $U \subset \delta^l$ of δ^l_c such that
$$h|_{U\times M} = \widetilde{h}|_{U\times M}$$
where h and $\widetilde{h}$ are the corresponding smooth maps $\delta^l \times M \to \mathbb{R}$ which realize the δ^l-families of cut tuples C and $\widetilde{C}$.
- We then also have an evident notion of *δ^l-families of cut $[m]$-tuples*: A *δ^l-family of cut $[m]$-tuples* is a collection
$$C := \{C_t := (C_{<(j,t)}, C_{=(j,t)}, C_{>(j,t)})_{j\in[m]} \mid t \in \delta^l\}$$
of cut $[m]$-tuples C_t such that for each $j \in [m]$ there exists a smooth function $h_j\colon \delta^l \times M \to \mathbb{R}$ so that $h_{(j,t)} := h_j(t, -)\colon M \to \mathbb{R}$ gives rise to the j-th cut in the cut-tuple C_t (as in Definition 8.24).

Remark 8.51. Some remarks are in order here:

- We have an ordering of δ^l-families of cuts, given by $C \leqslant C'$ if and only if $C_{\leqslant,t} \subset C'_{\leqslant,t}$ for all t in some neighborhood of $\delta^l_c \subset \delta^l$.

- The above definition gives rise to a simplicially enriched functor
$$\mathfrak{Cut}\colon \Delta^{\mathrm{op}} \times \mathfrak{FEmb}_d^{\mathrm{op}} \to \mathrm{sSet}$$
which maps a pair $([m], M \overset{p}{\twoheadrightarrow} U)$ to the simplicial set whose l-simplices are given by δ^l-families of cut $[m]$-tuples on $M \overset{p}{\twoheadrightarrow} U$. The face and degeneracy maps of the simplicial set $\mathfrak{Cut}([m], M \twoheadrightarrow U)$ take the j-th entry of some l-simplex given by a smooth map $h_j\colon \delta^l \times M \to \mathbb{R}$ to the compositions
$$\delta^{l-1} \times M \xrightarrow{|d^k| \times \mathrm{id}_M} \delta^l \times M \xrightarrow{\quad h_j \quad} \mathbb{R}$$
$$\delta^{l+1} \times M \xrightarrow{|s^k| \times \mathrm{id}_M} \delta^l \times M \xrightarrow{\quad h_j \quad} \mathbb{R}$$
where $|s^k|$ and $|d^k|$ are defined via (1). The simplicial structure map
$$\Delta([n],[m]) \times \mathfrak{FEmb}_d(M \overset{p}{\twoheadrightarrow} U, N \overset{q}{\twoheadrightarrow} V) \to \mathrm{Map}(\mathfrak{Cut}([m],q), \mathfrak{Cut}([n],p))$$
takes l-simplices on the LHS, say, the pair $(\omega, (f_t, F)_{t\in\delta^l})$, where ω is either a coface map $d^a\colon [n] \to [n+1]$ or a codegeneracy map $s^a\colon [n+1] \to [n]$, into the simplicial set whose k-simplices are given by homotopies
$$\Delta^k \times \mathfrak{Cut}([m], q) \to \mathfrak{Cut}([n], p)$$
Our associated induced map then has components
$$\begin{aligned} \Delta^k_r \times \mathfrak{Cut}([m], q)_r &\to \mathfrak{Cut}([n], p)_r \\ (g, (C_t)_{t\in\delta^r}) &\mapsto (\omega^\star f^{-1}_{|g|(t)}(C_t))_{t\in\delta^r} \end{aligned}$$
where $w^\star$, depending on whether $\omega = d^a$ or $\omega = s^a$, either removes the all the a-th cuts in the corresponding δ^r-family of cut tuples, or dublicates it.
- We may extend the above simplicially enriched functor $\mathfrak{Cut}$ to also include cuts in different simplicial directions: Interpret a subset $A \subset \{1, \ldots, d\}$ as a discrete category. We define the functor
$$\mathfrak{Cut}^A_{\pitchfork}\colon (\Delta^{\mathrm{op}})^A \times \mathrm{FEmb}_d^{\mathrm{op}} \to \mathrm{sSet}$$
which takes a pair $(\mathbf{m}, M \twoheadrightarrow U)$ to the simplicial set whose l-simplices are the subset of the product
$$\mathfrak{Cut}^A_{\pitchfork}(\mathbf{m}, M \twoheadrightarrow U)_l \subset \prod_{a\in A} \mathfrak{Cut}([m_a], M \twoheadrightarrow U)_l$$
consisting of those δ^l-families of cut tuples that satisfy the transversality condition as given in Definition 8.32. Face and degeneracy maps are analogous to what we defined before, and what $\mathfrak{Cut}^A_{\pitchfork}$ does to morphisms from $(\Delta^{\mathrm{op}})^A$ and FEmb_d is also in the same spirit as for $\mathfrak{Cut}$. More precisely, $\mathfrak{Cut}^A_{\pitchfork}$ is a subfunctor of the product of functors $\prod_{a\in A} \mathfrak{Cut}\colon (\Delta^{\mathrm{op}})^A \times \mathrm{FEmb}_d^{\mathrm{op}} \to \mathrm{sSet}$. For $A = \{1, \ldots, d\}$ being the whole set, we shall write
$$\mathfrak{Cut}_{\pitchfork} := \mathfrak{Cut}^{\{1,\ldots,d\}}_{\pitchfork}$$
whenever the dependency on the dimension d is evident.

Lemma 8.52. *Let* $(\mathbf{m}, \langle l\rangle, U) \in \Delta^{\times d} \times \Gamma \times \mathrm{Cart}$ *and fix a* d*-dimensional manifold* M *and a partition map* $P\colon M \times U \to \langle l\rangle$*. Let* C *be a* δ^k*-family of cut* $\mathbf{m}$*-grids on the projection* $p\colon M \times U \twoheadrightarrow U$*. Then for each* $u \in U$*, the union*
$$\mathrm{tot}(M, C, P)_u := \bigcup_{t\in\delta^k_c} \mathrm{core}(M, C, P)_{t,u}$$

where

$$\mathrm{core}(M,C,P)_{t,u} := p^{-1}\{u\} \cap \mathrm{core}(M,C_t,P)$$

is compact.

Proof. This is Lemma 4.3.3 in [16]. ☐

Based on the previous Lemma we have the following:

Definition 8.53. Let M, C and P be as in the above Lemma. We call $\mathrm{tot}(M,C,P)_u$ the *total core* of the δ^k-family of cut $\mathbf{m}$-grids at u. The union

$$\mathrm{tot}(M,C,P) := \bigcup_{u\in U} \mathrm{tot}(M,C,P)_u$$

is called the *total core* of C.

With that out of the way, we define a precursor to our bordism categories with isotopies:

Definition 8.54. Let $d \geqslant 0$. For fixed $(\mathbf{m}, \langle l\rangle, U) \in \Delta^{\times d} \times \Gamma \times \mathrm{Cart}$, the simplicial object $\mathfrak{B}(\mathbf{m}, \langle l\rangle, U)$ in Cat is given by the following data:

- The simplicial set of objects, whose vertices are called bordisms, is given by
$$\mathrm{Ob} := \coprod_{(M,P)} \mathfrak{Cut}_{\pitchfork}(\mathbf{m}, M\times U \twoheadrightarrow U)$$
where the simplicial set $\mathfrak{Cut}_{\pitchfork}$ was defined in Remark 8.51 and the coproduct ranges over all pairs (M,P) as given in Definition 8.32.
- In order to define the simplicial set of morphisms let us first define the following simplicial subset: For a pair $(M,P), (\widetilde{M},\widetilde{P})$ as in Definition 8.32, we define a simplicial subset
$$\mathfrak{MCut}((M,P),(\widetilde{M},\widetilde{P}))$$
of
$$\mathfrak{FEmb}_d(M\times U \twoheadrightarrow U, \widetilde{M}\times U \twoheadrightarrow U) \times \mathfrak{Cut}_{\pitchfork}(\mathbf{m}, M\times U \twoheadrightarrow U) \times \mathfrak{Cut}_{\pitchfork}(\mathbf{m}, \widetilde{M}\times U \twoheadrightarrow U)$$
where $\mathfrak{FEmb}_d(M\times U \twoheadrightarrow U, \widetilde{M}\times U \twoheadrightarrow U)$ is the Hom-object of the simplicial category $\mathfrak{FEmb}_d$ from Remark 8.5. An l-simplex of the simplicial subset $\mathfrak{MCut}((M,P),(\widetilde{M},\widetilde{P}))$ is a triple
$$(\psi = \psi^1 \times \mathrm{id}_U \colon \delta^l \times M \times U \to \widetilde{M}\times U,\ C,\ \widetilde{C})$$
We require that for all $t \in \delta^l$, the corresponding fiberwise embedding $\psi_t \colon M\times U \to \widetilde{M}\times U$ is, in particular, a fiberwise cut-respecting embedding in the sense of Definition 8.32 with respect to the corresponding cut $\mathbf{m}$-grids C_t and $\widetilde{C}_t$. In particular, we require that ψ is compatible with the maps P and $\widetilde{P}$ in the sense that the induced map on connected components
$$\psi_\star \colon \pi_0(M\times U) \cong \pi_0(M\times U\times \delta^l) \to \pi_0(\widetilde{M}\times U)$$
satisfies $\widetilde{P}\psi_\star = P$. Having all that, the simplicial set of morphisms is given by
$$\mathrm{Mor} := \coprod_{((M,P),(\widetilde{M},\widetilde{P}))} \mathfrak{MCut}((M,P),(\widetilde{M},\widetilde{P}))$$

- The source map $\mathrm{dom}_n\colon \mathrm{Mor}_n \to \mathrm{Ob}_n$ in the n-th simplicial layer takes a morphism $(((M,P),(\widetilde{M},\widetilde{P})),\psi,C,\widetilde{C})$ and maps it onto $((M,P),C)$. The target map $\mathrm{cod}_n\colon \mathrm{Mor}_n \to \mathrm{Ob}_n$ in the n-th simplicial layer takes a morphism $(((M,P),(\widetilde{M},\widetilde{P})),\psi,C,\widetilde{C})$ to $((\widetilde{M},\widetilde{P}),\widetilde{C})$.
- Composition of two morphisms $m_1 := (((M,P),(\widetilde{M},\widetilde{P})),\psi,C,\widetilde{C})$ and $m_2 := (((\widetilde{M},\widetilde{P}),(M',P')),\psi',\widetilde{C},C')$ in the n-th simplicial layer may be defined by setting
$$m_2 \circ m_1 := (((M,P),(M',P')),\Psi\colon \delta^n \times M \times U \to M' \times U, C, C')$$
where $\Psi^1_t := (\psi^{1\prime})_t \circ (\psi^1)_t$ for all $t \in \delta^n$.

Remark 8.55. Yet again we may extend the above definition to obtain a functor
$$\mathfrak{B}\colon (\Delta^{\times d} \times \Gamma \times \mathrm{Cart})^{\mathrm{op}} \to \mathrm{Cat}^{\Delta^{\mathrm{op}}}, \qquad (\mathbf{m},\langle l\rangle,U) \mapsto \mathfrak{B}(\mathbf{m},\langle l\rangle,U)$$

With that in our toolkit we may finally give a precise definition of the bordism categories with isotopies we are so interested in:

Definition 8.56. Fix $d \geqslant 0$. The *d-uple bordism category with isotopies* (without geometric structure) is the object $\mathfrak{Bord}_{d,\mathrm{uple}}$ in the (model) category (see Definition 7.98)
$$\mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{uple}}_{(\infty,d)}$$
given by the following composition of functors:
$$\begin{array}{ccc}
(\Delta^{\times d} \times \Gamma \times \mathrm{Cart})^{\mathrm{op}} & \xrightarrow{\ \mathfrak{B}\ } & \mathrm{Cat}^{\Delta^{\mathrm{op}}} \\
{\scriptstyle \mathfrak{Bord}_{(\infty,d),\mathrm{uple}}}\ \vdots & & \big\downarrow\ {\scriptstyle \mathfrak{N}} \\
\mathrm{sSet} & \xleftarrow[\ \mathrm{diag}\ ]{} & \mathrm{Psh}_\Delta(\Delta)
\end{array}$$

Example 8.57. A vertex in $\mathfrak{Bord}_{(\infty,d),\mathrm{uple}}(\mathbf{m},\langle l\rangle,U)$ is the same as a vertex in $\mathrm{Bord}_{(\infty,d),\mathrm{uple}}(\mathbf{m},\langle l\rangle,U)$, since δ^0-families of cut-grids are just single cut-grids. n-simplices in $\mathfrak{Bord}_{(\infty,d),\mathrm{uple}}(\mathbf{m},\langle l\rangle,U)$, on the other hand, are given by
$$\begin{aligned}\mathfrak{Bord}_{(\infty,d),\mathrm{uple}}(\mathbf{m},\langle l\rangle,U)_n &= \mathrm{diag}\circ\mathfrak{N}(\mathfrak{B}(m,\langle l\rangle,U))_n \\ &= \mathfrak{N}(\mathfrak{B}(\mathbf{m},\langle l\rangle,U)_n)_n\end{aligned}$$
In other words, such an n-simplex is given by n-many composable triplets
$$(\psi_j\colon \delta^n \times M_{j-1} \times U \to M_j \times U,\ C_{j-1},\ C_j)_{j=1}^n$$
where the ψ_j are δ^n-families of cut-respecting embeddings, while the C_j are δ^n-families of cut $\mathbf{m}$-grids.

Finally, it is time to add some more geometry to this flavor of a bordism category. For this, let us again start off with yet again another precursor.

Definition 8.58. Let $d \geqslant 0$ and let $\mathbf{S} \in \mathrm{Psh}_\Delta(\mathfrak{FEmb}_d)$ be a simplicial presheaf on the enriched site $\mathfrak{FEmb}_d$. For fixed $(\mathbf{m},\langle l\rangle,U) \in \Delta^{\times d} \times \Gamma \times \mathrm{Cart}$, the simplicial object $\mathfrak{B}^{\mathbf{S}}(\mathbf{m},\langle l\rangle,U)$ in Cat is given by the following data:

- The simplicial set of objects is given by
$$\mathrm{Ob} := \coprod_{(M,P)} \mathfrak{Cut}_{\pitchfork}(\mathbf{m}, M \times U \twoheadrightarrow U) \times \mathbf{S}(M \times U \twoheadrightarrow U)$$
where the coproduct ranges over the pairs (M,P) from Definition 8.32 and $\mathfrak{Cut}_{\pitchfork}$ was defined in Remark 8.51.

- The simplicial set of morphisms is given by
$$\mathrm{Mor} := \coprod_{((M,P),(\widetilde{M},\widetilde{P}))} \mathfrak{MCut}((M,P),(\widetilde{M},\widetilde{P})) \times \mathbf{S}(\widetilde{M} \times U \twoheadrightarrow U)$$
where the coproduct ranges over pairs $((M,P),(\widetilde{M},\widetilde{P}))$ as given in Definition 8.32 and $\mathfrak{MCut}$ was defined in Definition 8.54.
- The source map $\mathrm{dom}_n \colon \mathrm{Mor}_n \to \mathrm{Ob}_n$ in the n-th simplicial layer sends a morphism
$$(((M,P),(\widetilde{M},\widetilde{P})),\psi,C,\widetilde{C},s \in \mathbf{S}(\widetilde{M} \times U \twoheadrightarrow U)_n)$$
to
$$((M,P),\mathbf{S}(\psi)(s) \in \mathbf{S}(M \times U \twoheadrightarrow U))$$
where $\mathbf{S}(\psi)(s)$ is provided by means of the enriched presheaf structure maps
$$\mathfrak{FEmb}_d(M \times U \twoheadrightarrow U, \widetilde{M} \times U \twoheadrightarrow U) \times \mathbf{S}(\widetilde{M} \times U \twoheadrightarrow U) \longrightarrow \mathbf{S}(M \times U \twoheadrightarrow U)$$
The target map $\mathrm{cod}_n \colon \mathrm{Mor}_n \to \mathrm{Ob}_n$ in the n-th simplicial layer sends a morphism
$$(((M,P),(\widetilde{M},\widetilde{P})),\psi,C,\widetilde{C},s \in \mathbf{S}(\widetilde{M} \times U \twoheadrightarrow U)_n)$$
to
$$((\widetilde{M},\widetilde{P}),s)$$
- Composition of two composable morphisms
$$m_1 := (((M,P),(\widetilde{M},\widetilde{P})),\psi,C,\widetilde{C},s \in \mathbf{S}(\widetilde{M} \times U \twoheadrightarrow U)_n)$$
$$m_2 := ((\widetilde{M},\widetilde{P}),(M',P')),\psi',\widetilde{C},C',s' \in \mathbf{S}(M' \times U \twoheadrightarrow U)_n)$$
is given by
$$(((M,P),(M',P')),\Psi,C,C',s')$$
where $\Psi^1_t := (\psi^{1\prime})_t \circ (\psi^1)_t$ for all $t \in \delta^n$.

Remark 8.59. Again, the previous definition collects into a functor:
$$\mathfrak{B}^{\mathbf{S}} \colon (\Delta^{\times d} \times \mathit{\Gamma} \times \mathrm{Cart})^{\mathrm{op}} \to \mathrm{Cat}^{\Delta^{\mathrm{op}}}, \qquad (\mathbf{m},\langle l\rangle,U) \mapsto \mathfrak{B}^{\mathbf{S}}(\mathbf{m},\langle l\rangle,U)$$

Definition 8.60. Fix $d \geqslant 0$ and $\mathbf{S} \in \mathrm{Psh}_\Delta(\mathfrak{FEmb}_d)$. The *d-uple bordism category with isotopies and geometric structure* $\mathbf{S}$ is the object $\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$ in the (model) category (see Definition 7.98)
$$\mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{uple}}_{(\infty,d)}$$
given by the following composition of functors:
$$\begin{array}{ccc} (\Delta^{\times d} \times \mathit{\Gamma} \times \mathrm{Cart})^{\mathrm{op}} & \xrightarrow{\mathfrak{B}^{\mathbf{S}}} & \mathrm{Cat}^{\Delta^{\mathrm{op}}} \\ \Big\downarrow{\scriptstyle \mathfrak{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}} & & \Big\downarrow{\scriptstyle \mathfrak{N}} \\ \mathrm{sSet} & \xleftarrow[\mathrm{diag}]{} & \mathrm{Psh}_\Delta(\Delta) \end{array}$$

Example 8.61. Consider the 1-dimensional geometric structure with isotopies $\mathfrak{Riem}_1^{\mathfrak{f}}$ from Example 8.18. The datum of a vertex in $\mathfrak{Bord}_{(\infty,1)}^{\mathfrak{Riem}_1^{\mathfrak{f}}}([1],\langle 1\rangle,\mathbb{R}^0)$ is the same as a vertex in $\mathrm{Bord}_{(\infty,1)}^{\mathrm{Riem}_1^{\mathfrak{f}}}([1],\langle 1\rangle,\mathbb{R}^0)$, that is, a triple $(M,C,\mathfrak{m})$ where M is a 1-dimensional manifold, C is a cut [1]-tuple on M and $\mathfrak{m}$ is a Riemannian metric on M. On the other hand, a 1-simplex in $\mathfrak{Bord}_{(\infty,1)}^{\mathfrak{Riem}_1^{\mathfrak{f}}}([1],\langle 1\rangle,\mathbb{R}^0)$ is given by a tuple

$$(\psi\colon M\times\delta^1\to\widetilde{M},\ C,\widetilde{C},(\mathfrak{m}_t)_{t\in\delta^1})$$

where ψ is a δ^1-family of cut-respecting embeddings $M\to\widetilde{M}$ with respect to two δ^1-families of cut [1]-tuples C and $\widetilde{C}$ for M and N, respectively. Applying the face maps d_0 and d_1 to the above tuple yields

$$(\widetilde{M},\widetilde{C}_{t=0},\widetilde{\mathfrak{m}}_{t=0}),\qquad (M,(\psi^\star\widetilde{C})_{t=1},\psi^\star\widetilde{\mathfrak{m}}_{t=1})$$

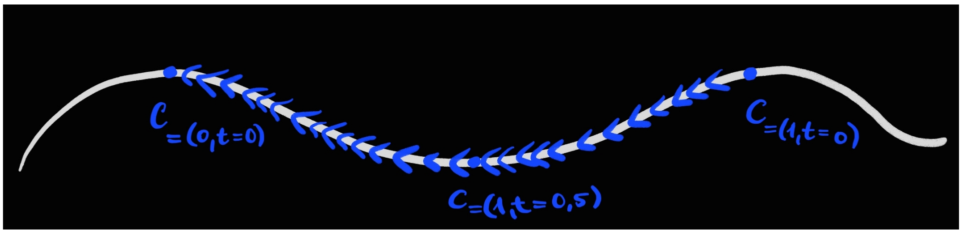
One such 1-simplex could for example be given by

$$(\psi,\ C,\ C,\ (\mathfrak{m}_t)_{t\in\delta^1}:=(\mathfrak{m})_{t\in\delta^1})$$

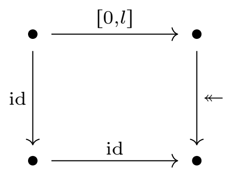
where

$$\psi\colon M\times\delta^1\to\widetilde{M},\qquad \psi(m,t):=m$$

and $\mathfrak{m}$ is a metric on M, while $C=(C_t)_{t\in\delta^1}$ is the δ^1-family of cut [1]-tuples given by keeping the cut locus $C_{=(0,t=0)}$ fixed and moving the cut locus $C_{=(1,t=0)}$ to $C_{=(0,t=0)}$:



Remark 8.62. In the above example it might seem as if the explicit 1-simplex $(\psi,C,C,(\mathfrak{m})_{t\in\delta^1})$ from the above example (or any more general 1-simplex) collapses the data of the Riemannian length. However, this is not the case as the δ^1-family records the information of the entire family of Riemannian lenghts. This is best showcased by referring to the Segal formalism:



In the above, $[0,l]$ denotes the manifold with Riemannian length, the arrow $\twoheadleftarrow$ denotes the isotopy of points which moves the endpoint in $[0,l]$ to the starting point.

Example 8.63. Let us consider

$$\mathfrak{Bord}_{(\infty,1)}^{\text{\mathfrak{s}}(\mathbb{R}\times U\twoheadrightarrow U)}([0],\langle 1\rangle,V)$$

An l-simplex in the above simplicial set is given by a composable l-tuple of morphisms in the category $\mathfrak{B}^{\text{\mathfrak{s}}(\mathbb{R}^d\times U\twoheadrightarrow U)})_l$:

$$(\psi_j\colon\delta^l\times M_{j-1}\times V\to M_j\times V,\ C_{j-1},\ C_j,\ (f_j,F_j))_{j=1}^l$$

where the ψ_j are δ^l-families of cut-respecting embeddings, $M_j \subset \mathbb{R}$ are open submanifolds, the C_j are δ^l-families of cuts, while

$$(f_j, F_j) \in \mathfrak{FEmb}_d(M_j \times V \twoheadrightarrow V, \mathbb{R} \times U \twoheadrightarrow U)_l$$

is a δ^l-family of fiberwise embeddings. In other words, $f_j \colon \delta^l \times M_j \times V \to \mathbb{R} \times U$ and $F_j \colon V \to U$ are smooth maps such that

$$\begin{array}{ccc} M_j \times V & \xrightarrow{f_j(t,-)} & \mathbb{R} \times U \\ \big\downarrow & & \big\downarrow \\ V & \xrightarrow[F_j]{} & U \end{array}$$

represents a morphism in FEmb_1 for all $t \in \delta^l$. So $f_j(t,-)$ embeds the fiber $M_j \times \{v\}$ into $\mathbb{R} \times \{F(v)\}$.

Example 8.64. Consider the 0-dimensional bordism category $\mathfrak{Bord}_{(\infty,0)}^{\mathfrak{C}^\infty(-,\mathfrak{X})}$ endowed with the geometric structure from Example 8.19. We note that, since $d = 0$, vertices in $\mathfrak{Bord}_{(\infty,0)}^{\mathfrak{C}^\infty(-,\mathfrak{X})}(\langle 1\rangle, U)$ are elements in the set

$$\mathfrak{C}^\infty(M \times U, \mathfrak{X})_0 \cong \prod_M \mathscr{C}^\infty(U, \mathfrak{X})$$

where M is a 0-dimensional manifold (a disjoint union of points). An l-simplex in $\mathrm{Bord}_{(\infty,0)}^{\mathfrak{C}^\infty(-,\mathfrak{X})}(\langle 1\rangle, U)$ is given by an l-tuple

$$(\psi_j \colon \delta^l \times M_{j-1} \times U \to M_j \times U,\ \alpha)_{j=1}^l$$

where ψ is a diffeomorphism, while $\alpha \colon \delta^l \times N \times U \to \mathfrak{X}$ is a smooth map.

Remark 8.65. All cylinders in a fibrant replacement of $\mathfrak{Bord}_{(\infty,d)}^{\mathbf{S}}$ are invertible. For example, for $d = 1$ consider an interval with source cut-tuple $C_0 := (C_{<0}, C_{=0}, C_{>0})$ and target cut-tuple $C_1 := (C_{<1}, C_{=1}, C_{>1})$. The source cut-tuple is induced by a smooth map $h \colon M \to \mathbb{R}$. Consider the additive inverse $-h \colon M \to \mathbb{R}$ which results in a new cut

$$D_0 := (D_{<0} := C_{>0}, D_{=0} := C_{=0}, D_{>0} := C_{<0})$$

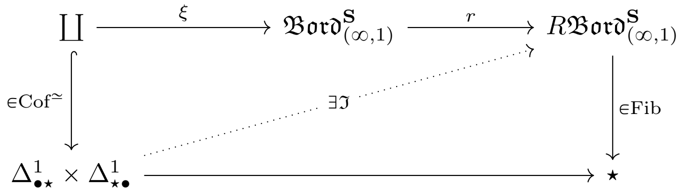
Now consider the isotopy $\mathfrak{i}$ of points (an element of $\mathfrak{Bord}_{(\infty,d)}^{\mathbf{S}}(\mathbf{0})_1$) which transports the cut $D_{=0}$ to the source cut $C_{=0}$. All this results in a diagram

$$\begin{array}{ccc} \bullet & & \bullet \\ {\scriptstyle \mathfrak{i}}\big\downarrow & & \big\downarrow{\scriptstyle (s_0)_{\star\bullet}} \\ \bullet & \xrightarrow[(s_0)_{\bullet\star}]{} & \bullet \end{array}$$

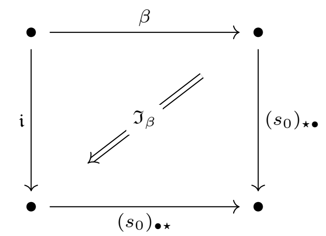
(where $(s_0)_{\bullet\star}, (s_0)_{\star\bullet}$ are the respective identities in the different simplicial directions) which may be viewed as a morphism of simplicial presheaves

$$\xi \in \mathrm{Hom}(\coprod, \mathfrak{Bord}_{(\infty,d)}^{\mathbf{S}})$$

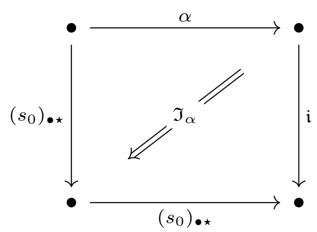
where $\coprod$ does not denote the coproduct, but rather the glueing of corresponding copies of $\Delta^1_{\bullet\star}$ and $\Delta^1_{\star\bullet}$. Now let R be some fibrant replacement functor, then we have the lifting problem



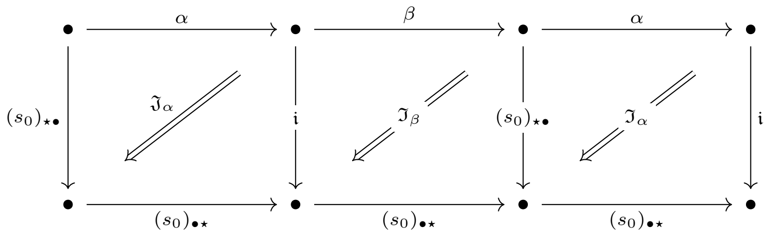
which has a solution $\mathfrak{I}_\beta$ that may be depicted by



The β thus obtained will be an inverse to the initial bordism α. Indeed, α itself gives rise to a square



We can then glue these two squares to obtain



Considering the glueing of the two squares to the right we obtain $\alpha\beta \simeq \mathrm{id}$, while considering the glueing of the two squares to the left results in $\beta\alpha \simeq \mathrm{id}$.

8.5. **Globular smooth Bordism Categories.** We defined essentially two different d-uple-bordism categories with geometric structures $\mathbf{S}$ denoted by $\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$ and $\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$. From these two, we shall extract globular bordism categories $\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{glob}}$ and $\mathfrak{Bord}^{S}_{(\infty,d),\mathrm{glob}}$.

Definition 8.66. Let $d \geqslant 0$ and let $\mathbf{S} \in \mathrm{Psh}_\Delta(\mathrm{FEmb}_d)$ or $\mathbf{S} \in \mathrm{Psh}_\Delta(\mathfrak{FEmb}_d)$ (depending on the bordism category) be a geometric structure.

- The *globular bordism category* $\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$ is defined as follows: Let
$$\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{glob}} \subset \mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$$
be the subobject whose value at $(\mathbf{m}, \langle l \rangle, U)$ with $\mathbf{m} = ([m_1], \ldots, [m_d])$ is the diagonal of the nerve of the simplicial subobject $\mathrm{B}^{\mathbf{S}}_{\mathrm{glob}}$ whose simplicial set of objects is given by taking only those summands in Definition 8.41 that are indexed by triples (M, C, P) satisfying the following property:

– If

$$(M,C,P) \in \mathrm{Bord}_{(\infty,d),\mathrm{uple}}(([m_1],\ldots,[m_{i-1}],[0],[m_{i+1}],\ldots,[m_d]),\langle l\rangle,U)$$

for some $1 \leqslant i \leqslant d$, then (M,C,P) lies in the same connected component as a simplicial degeneration of an object in

$$\mathrm{Bord}_{(\infty,d),\mathrm{uple}}(([m_1],\ldots,[m_{i-1}],[0],\ldots,[0]),\langle l\rangle,U)$$

in the simplicial directions $i+1, i+2, \ldots, d$.

- The *globular bordism category with isotopies* $\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{glob}}$ is defined as follows: Let

$$\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{glob}} \subset \mathfrak{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$$

be the subobject whose value at $(\mathbf{m},\langle l\rangle,U)$ with $\mathbf{m} = ([m_1],\ldots,[m_d])$ is the diagonal of the nerve of the simplicial subobject $\mathfrak{B}^{\mathbf{S}}_{\mathrm{glob}}$ whose simplicial set of objects is given by taking only those summands in Definition 8.58 that are indexed by triples (M,P) satisfying the following property: If

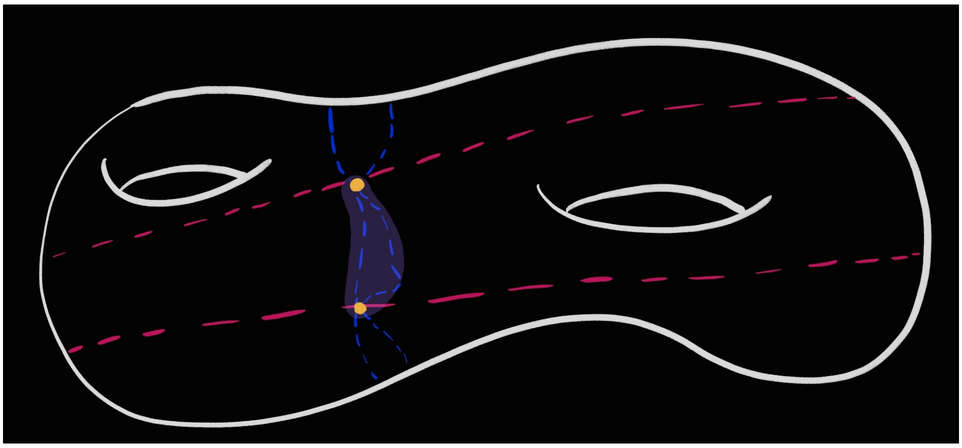
$$(M,C,P) \in \mathfrak{Bord}_{(\infty,d),\mathrm{uple}}(([m_1],\ldots,[m_{i-1}],[0],[m_{i+1}],\ldots,[m_d]),\langle l\rangle,U)$$

where C is a δ^0-family of cut $\mathbf{m}$-tuples (that is, C is one cut $\mathbf{m}$-grid) for some $1 \leqslant i \leqslant d$, then (M,C,P) lies in the same connected component as a simplicial degeneration of an object in

$$\mathfrak{Bord}_{(\infty,d),\mathrm{uple}}(([m_1],\ldots,[m_{i-1}],[0],\ldots,[0]),\langle l\rangle,U)$$

in the simplicial directions $i+1, i+2, \ldots, d$.

Example 8.67. Let $d = 2$, then the following image depicts a cut tuple in the globular bordism category in bidegree $([1],[1])$, that is, a vertex of $\mathfrak{Bord}_{(\infty,2)}(([1],[1]),\langle 1\rangle,\mathbb{R}^0)$:



Remark 8.68. In general the objects

$$\mathrm{Bord}^{\mathbf{S}}_{(\infty,d)}, \qquad \mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)}$$

(globular or mutliple) are not fibrant in the respective model category $\mathscr{C}^\infty\mathrm{Cat}^{\otimes}_{(\infty,d)}$ (this denotes either the multiple injective model structure or the globular one). This is no problem however, since in the end we will be interested solely in derived Hom-spaces. Since any object in $\mathscr{C}^\infty\mathrm{Cat}^{\otimes}_{(\infty,d)}$ is cofibrant, we do not need to (co)fibrantly replace $\mathrm{Bord}^{\mathbf{S}}_{(\infty,d)}$ or $\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)}$ in the domain slot of our derived Homs.

8.6. **Functoriality of Bordism Categories.** In the previous few pages we have witnessed that there are assignments of objects

$$\mathrm{Ob}(\mathrm{Psh}_\Delta(\mathrm{FEmb}_d)) \to \mathrm{Ob}(\mathscr{C}^\infty\mathrm{Cat}^{\otimes}_{(\infty,d)}), \qquad \mathbf{S} \mapsto \mathrm{Bord}^{\mathbf{S}}_{(\infty,d)}$$
$$\mathrm{Ob}(\mathrm{Psh}_\Delta(\mathfrak{FEmb}_d)) \to \mathrm{Ob}(\mathscr{C}^\infty\mathrm{Cat}^{\otimes}_{(\infty,d)}), \qquad \mathbf{S} \mapsto \mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)}$$

where $\mathscr{C}^\infty\mathrm{Cat}^{\otimes}_{(\infty,d)}$ could stand both for the globular injective model structure and the multiple injective model structure. It is readily observed that these maps extend to yield a functor between the respective categories. For example, the functor

$$\mathrm{Bord}^{(-)}_{(\infty,d),\mathrm{uple}} \colon \mathrm{Psh}_\Delta(\mathrm{FEmb}_d) \to \mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{uple}}_{(\infty,d)}$$

does the obvious things to objects:

$$\mathbf{S} \mapsto \mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$$

A morphism $\mathbf{S} \to \mathbf{T}$ in $\mathrm{Psh}_\Delta(\mathrm{FEmb}_d)$ is mapped to the morphism

$$\mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}} \to \mathrm{Bord}^{\mathbf{T}}_{(\infty,d),\mathrm{uple}}$$

obtained from the induced maps

$$\coprod_{(M,C,P)} \mathbf{S}(M \times U \twoheadrightarrow U) \longrightarrow \coprod_{(M,C,P)} \mathbf{T}(M \times U \twoheadrightarrow U)$$

$$\coprod_{(M,C,P)\xrightarrow{\psi}(\widetilde{M},\widetilde{C},\widetilde{P})} \mathbf{S}(\widetilde{M} \times U \twoheadrightarrow U) \longrightarrow \coprod_{(M,C,P)\xrightarrow{\psi}(\widetilde{M},\widetilde{C},\widetilde{P})} \mathbf{T}(\widetilde{M} \times U \twoheadrightarrow U)$$

by applying $\mathrm{diag} \circ \mathfrak{N}$. We then have the following:

Proposition 8.69. *Let $d \geqslant 0$. The functors*

$$\mathrm{Bord}^{(-)}_{(\infty,d),\mathrm{uple}} \colon \mathrm{Psh}_\Delta(\mathrm{FEmb}_d)_{\mathrm{inj}} \to \mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{uple}}_{(\infty,d)}, \qquad \mathbf{S} \mapsto \mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$$
$$\mathrm{Bord}^{(-)}_{(\infty,d),\mathrm{glob}} \colon \mathrm{Psh}_\Delta(\mathrm{FEmb}_d)_{\mathrm{inj}} \to \mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{glob}}_{(\infty,d)}, \qquad \mathbf{S} \mapsto \mathrm{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{glob}}$$

are left Quillen functors that preserve all weak equivalences. In particular, they are homotopy continuous. Similarily, let $\mathrm{Psh}_\Delta(\mathfrak{FEmb}_d)_{\mathrm{inj}}$ denote the injective model structure on enriched presheaves. The functors

$$\mathfrak{Bord}^{(-)}_{(\infty,d),\mathrm{uple}} \colon \mathrm{Psh}_\Delta(\mathfrak{FEmb}_d)_{\mathrm{inj}} \to \mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{uple}}_{(\infty,d)}, \qquad \mathbf{S} \mapsto \mathfrak{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{uple}}$$
$$\mathfrak{Bord}^{(-)}_{(\infty,d),\mathrm{glob}} \colon \mathrm{Psh}_\Delta(\mathfrak{FEmb}_d)_{\mathrm{inj}} \to \mathscr{C}^\infty\mathrm{Cat}^{\otimes,\mathrm{glob}}_{(\infty,d)}, \qquad \mathbf{S} \mapsto \mathfrak{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{glob}}$$

are left Quillen functors that preserve all weak equivalences. In particular, they are homotopy cocontinuous.

Proof. Let us simply write $\mathrm{Bord}_{(\infty,d)}$ for both variants (globular and multiple). By definition $\mathbf{S} \mapsto \mathrm{Bord}^{\mathbf{S}}_{(\infty,d)}$ preserves monomorphisms and it maps weak equivalences to weak equivalences. Of course this functor is a left adjoint: its right adjoint is given by sending $X \in \mathscr{C}^\infty\mathrm{Cat}^{\otimes}_{(\infty,d)}$ to the simplicial presheaf

$$(M \twoheadrightarrow U) \mapsto \mathrm{Map}(\mathrm{Bord}^{j(M\twoheadrightarrow U)}_{(\infty,d)}, X)$$

The same argument works for $\mathfrak{Bord}_{(\infty,d)}$. □

More generally, the main result in [16] is the following:

Theorem 8.70. *The functors*

$$\mathrm{Psh}_\Delta(\mathrm{FEmb}_d)_{\check{\mathrm{C}}\mathrm{ech}} \to \mathscr{C}^\infty\mathrm{Cat}^{\otimes}_{(\infty,d)}, \qquad \mathbf{S} \mapsto \mathrm{Bord}^{\mathbf{S}}_{(\infty,d)}$$
$$\mathrm{Psh}_\Delta(\mathfrak{FEmb}_d)_{\check{\mathrm{C}}\mathrm{ech}} \to \mathscr{C}^\infty\mathrm{Cat}^{\otimes}_{(\infty,d)}, \qquad \mathbf{S} \mapsto \mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)}$$

are left Quillen functors. In particular, they are $(\infty,1)$*-cosheaves, i.e., they preserve homotopy colimits.*

8.7. **Symmetric Monoidal Structure of Smooth Bordism Categories.** We shall investigate the symmetric monoidal structure of $\mathfrak{g}\mathrm{Bord}_{(\infty,d),\mathrm{glob}}$. The other variants of bordism categories are analogous.

Proposition 8.71. *The functor* $\mathfrak{g}\mathrm{Bord}_{(\infty,d),\mathrm{glob}}$ *satisfies Segal's special* Γ*-condition.*

Proof. Let $\mathscr{C} := \mathfrak{g}\mathrm{Bord}_{(\infty,d),\mathrm{glob}}$. We have to show that the induced morphism

$$\mathscr{C}\langle l\rangle \xrightarrow{(\delta^!_1,\ldots,\delta^!_l)} \mathscr{C}\langle 1\rangle^l$$

where $\delta^!_j := \mathscr{C}\delta_j$, is an objectwise weak equivalence. We first note that the single morphism $\delta^!_i$ takes a bordism and forgets about all the information that is not contained in the i-th slot $P^{-1}\{i\}$. In other words, applying $\delta^!_i$ forgets about all the other connected components except the i-th. The most natural candidate for a homotopy inverse is then, unsurprisingly so, the l-fold disjoint union

$$\coprod^l : \mathscr{C}\langle 1\rangle^l \to \mathscr{C}\langle l\rangle, \qquad ((M_1,C_1,P_1),\ldots,(M_l,C_l,P_l)) \mapsto (\coprod_{i=1}^l M_i, \coprod_{i=1}^l C_i, \coprod_{i=1}^l P_i)$$

where $\coprod M_i$ is the manifold obtained by taking the disjoint union over all M_i, while $\coprod C_i$ is simply the cut-tuple obtained from the respective cut-functions for each $M_i \times U$ by just taking their disjoint union. The map of connected components $\coprod P_i$ is then defined by means of

$$\coprod P_i\colon \coprod M_i \times U \to \langle l\rangle, \qquad M_i \times U \ni (m_i,u) \mapsto \begin{cases} i, & \text{if } P_i(m_i,u) = 1 \\ \star, & \text{else} \end{cases}$$

We then note that $\coprod^l$ actually yields a genuine inverse for $(\delta^!_1,\ldots,\delta^!_l)$ (recall that we work in $\mathfrak{g}\mathrm{Bord}$, hence the Segal condition is verified. □

Remark 8.72. Of course the above Proposition is wrong if we talked about $\mathfrak{g}\mathrm{Bord}^{\mathbf{S}}_{(\infty,d)}$, $\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)}$ etc. for some non-trivial geometric structure. However, after passing to some fibrant replacement of the corresponding bordism category everything works out again.

We recall that the tensor ∞-functor for a symmetric monoidal ∞-category was constructed by taking a weak inverse of the map $(\delta^!_1,\delta^!_2)$, so in our case $\coprod\colon \mathscr{C}\langle 1\rangle^2 \to \mathscr{C}\langle 2\rangle$, and then by precomposing this weak inverse with $\varphi^!$, where $\varphi\colon \langle 2\rangle \to \langle 1\rangle, 1,2 \mapsto 1$. By means of that, we obtain $\otimes$:

$$\begin{array}{ccc} \mathscr{C}\langle 1\rangle^2 & \xrightarrow{\ \coprod\ } & \mathscr{C}\langle 2\rangle \\ & \overset{\otimes}{\searrow} & \downarrow{\scriptstyle \varphi^!} \\ & & \mathscr{C}\langle 1\rangle \end{array}$$

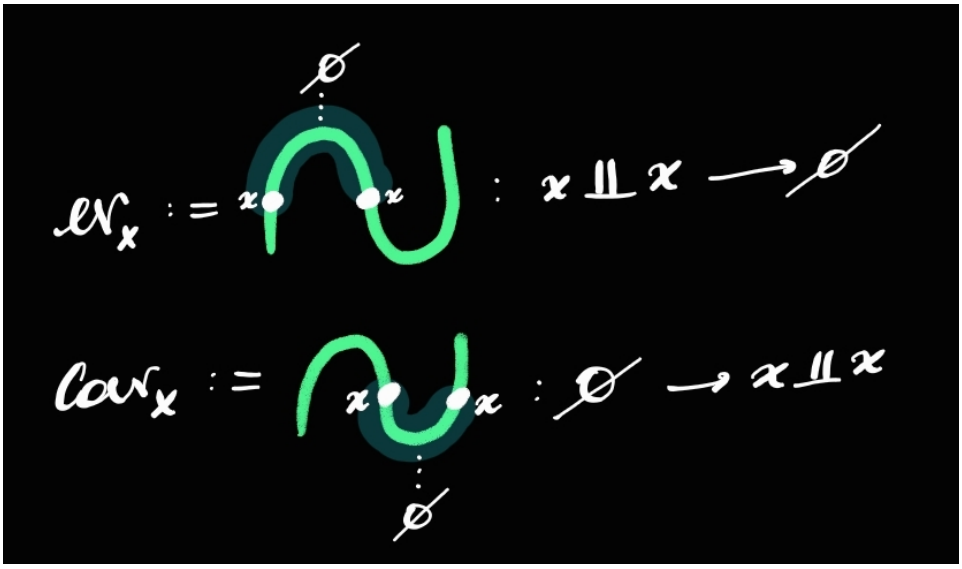
Since $\otimes$ is essentially taking the disjoint union itself with the difference that we collect all connected components in the slot $1 \in \langle 1\rangle$ (this is what postcomposing $\coprod$ with $\varphi^!$ does after all), we shall again write $\coprod = \otimes$. For completeness, let us consider the remaining structure maps for symmetric monoidality. For this, let us take the invesigate the variant $\mathfrak{Bord}_{(\infty,1)}$, as this allows us to also talk about duality information. We use the notation from the proof of Propostion 7.89. The unit object for the monoidal structure is of course given by the empty set (in fact, by any triplet (M, C, P) where $P\colon M \times U \to \langle 1\rangle$ is a partition function mapping everything to $\star$). The left and right unitors are essentially identities

$$\begin{array}{ccccc}
\mathfrak{Bord}_{(\infty,d)}\langle 1\rangle & \xrightarrow{\iota_i} & \mathfrak{Bord}_{(\infty,d)}\langle 2\rangle & \xrightarrow{(\delta_1^!,\delta_2^!)} & \mathfrak{Bord}_{(\infty,d)}\langle 1\rangle \times \mathfrak{Bord}_{(\infty,d)}\langle 1\rangle \\
 & \diagdown\!\!\diagdown & & & \downarrow{\scriptstyle \amalg} \\
 & & & & \mathfrak{Bord}_{(\infty,d)}\langle 2\rangle \\
 & & & & \downarrow{\scriptstyle \varphi^!} \\
 & & & & \mathfrak{Bord}_{(\infty,d)}\langle 1\rangle
\end{array}$$

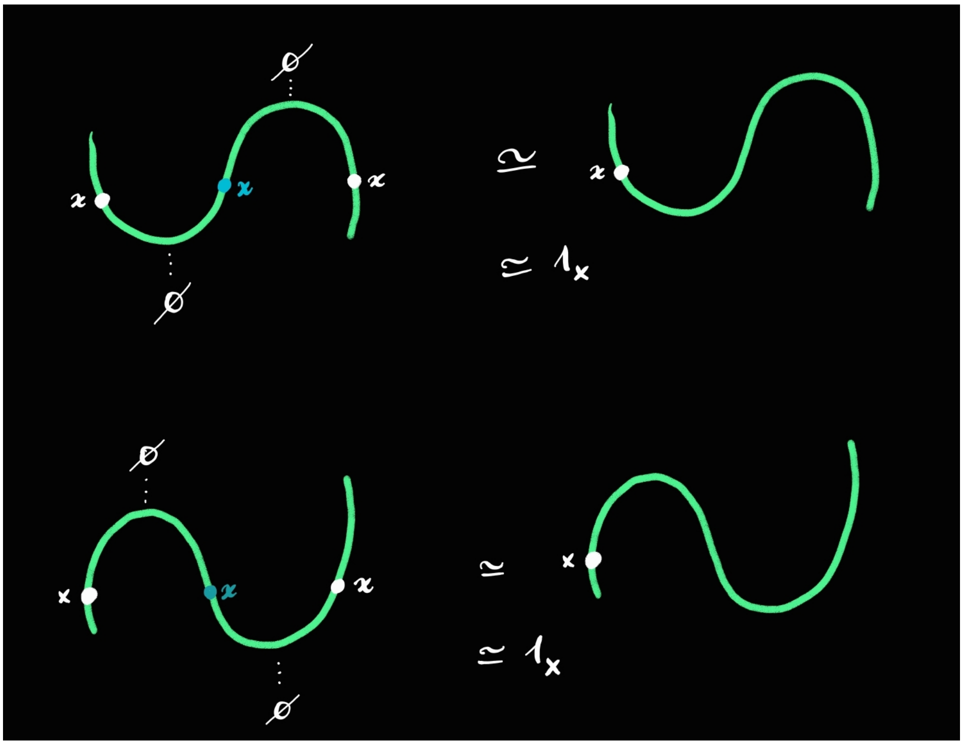
Analogously, the braiding is an identity:

$$\begin{array}{ccccc}
\mathfrak{Bord}_{(\infty,d)}\langle 1\rangle \times \mathfrak{Bord}_{(\infty,d)}\langle 1\rangle & \xrightarrow{\amalg} & \mathfrak{Bord}_{(\infty,d)}\langle 2\rangle & \xrightarrow{t^!} & \mathfrak{Bord}_{(\infty,d)}\langle 2\rangle \\
 & \diagdown\!\!\diagdown & & & \downarrow{\scriptstyle (\delta_1^!,\delta_2^!)} \\
 & & & & \mathfrak{Bord}_{(\infty,d)}\langle 1\rangle \times \mathfrak{Bord}_{(\infty,d)}\langle 1\rangle \\
 & & & & \downarrow{\scriptstyle \amalg} \\
 & & & & \mathfrak{Bord}_{(\infty,d)}\langle 2\rangle \\
 & & & & \downarrow{\scriptstyle \varphi^!} \\
 & & & & \mathfrak{Bord}_{(\infty,d)}\langle 1\rangle
\end{array}$$

And similarily the associator is just an identity. Evaluation and coevaluation maps for some point x may be given by:



But then the triangle identities for the duality data are trivially satisfied up to homotopy (isotopy):



9. Smooth Functorial Field Theories

> Circumstantial evidence is a very tricky thing. It may seem to point very straight to one thing, but if you shift your own point of view a little, you may find it pointing in an equally uncompromising manner to something entirely different.
>
> Sherlock Holmes, The Boscombe Valley Mystery

This chapter is based on the papers [24] and [17], as well as on discussions with Dmitri Pavlov.

9.1. **The Topological Cobordism Hypothesis.** Let us first delve into the formulation of the cobordism hypothesis as given by Lurie in [24]. Lurie also constructed an (∞, d)-category $\mathscr{B}\mathrm{ord}_{(\infty,d)}$ of bordisms. We will not review his construction in detail, but we will give the following sketch from [24]:

Definition 9.1. Let d be a nonnegative integer. The globular (∞, d)-category $\mathscr{B}\mathrm{ord}_{(\infty,d)}$ is described informally as follows:

- The objects of $\mathscr{B}\mathrm{ord}_{(\infty,d)}$ are 0-manifolds.
- The 1-morphisms of $\mathscr{B}\mathrm{ord}_{(\infty,d)}$ are bordisms between 0-manifolds.
- The 2-morphisms of $\mathscr{B}\mathrm{ord}_{(\infty,d)}$ are bordisms between bordisms between 0-manifolds.
- The n-morphisms of $\mathscr{B}\mathrm{ord}_{(\infty,d)}$ are bordisms between bordisms between ... between bordisms between 0-manifolds (in other words, n-manifolds with corners).
- The $(d + 1)$-morphisms of $\mathscr{B}\mathrm{ord}_{(\infty,d)}$ are diffeomorphisms (which reducce to the identity on the boundaries of the relevant manifolds).
- The $(d + 2)$-morphisms of $\mathscr{B}\mathrm{ord}_{(\infty,d)}$ are isotopies of diffeomorphisms.
- ...

In particular, we may also consider the variants ${}^{\mathrm{L}}\mathrm{Bord}^{\mathrm{fr}}_{(\infty,d)}$ and ${}^{\mathrm{L}}\mathrm{Bord}^{\mathrm{or}}_{(\infty,d)}$ where manifolds come equipped with d-framings and orientations, respectively.

Let $\mathscr{C}$ be a symmetric monoidal (∞, d)-category. In Lurie's setting a *$\mathscr{C}$-valued d-dimensional (framed) topological quantum field theory* is a symmetric monoidal ∞-functor

$$\mathfrak{F}\colon \mathscr{B}\mathrm{ord}^{\mathrm{fr}}_{(\infty,d)} \to \mathscr{C}$$

By means of the corresponding derived internal hom, we obtain a symmetric monoidal (∞, d)-category of topological quantum field theories, denoted by

$$\mathrm{Fun}^{\otimes}(\mathscr{B}\mathrm{ord}^{\mathrm{fr}}_{(\infty,d)}, \mathscr{C})$$

The statement of the most prominent variant of the *topological cobordism hypothesis* is then the following:

Theorem 9.2 ([24]). *Let $\mathscr{C}$ be a symmetric monoidal (∞, d)-category with duals. Then the evaluation functor $\mathfrak{F} \mapsto \mathfrak{F}(\mathrm{pt})$ induces an equivalence*

$$\mathrm{Fun}^{\otimes}(\mathscr{B}\mathrm{ord}^{\mathrm{fr}}_{(\infty,d)}, \mathscr{C}) \to \mathscr{C}^{\times}$$

where $(-)^{\times}\colon \mathrm{Cat}^{\otimes,\dagger}_{\infty,d} \to \mathrm{Grpd}^{\otimes}_{\infty}$ is the functor which extracts the maximal ∞-subgroupoid (see 7.136). In particular, $\mathrm{Fun}^{\otimes}(\mathscr{B}\mathrm{ord}^{\mathrm{fr}}_{(\infty,d)}, \mathscr{C})$ is an $(\infty, 0)$-category.

Remark 9.3. It is no restriction to assume that $\mathscr{C}$ has duals in the above theorem. In fact, for any symmetric monoidal (∞, d)-category $\mathscr{C}$, the canonical map

$$\mathrm{Fun}^{\otimes}(\mathscr{B}\mathrm{ord}^{\mathrm{fr}}_{(\infty,d)}, \mathscr{C}^{\mathrm{fd}}) \to \mathrm{Fun}^{\otimes}(\mathscr{B}\mathrm{ord}^{\mathrm{fr}}_{(\infty,d)}, \mathscr{C})$$

is an equvialence of (∞, d)-categories. Combining this observation with the above Theorem yields

$$\mathrm{Fun}^{\otimes}(\mathscr{B}\mathrm{ord}^{\mathrm{fr}}_{(\infty,d)}, \mathscr{C}) \simeq (\mathscr{C}^{\mathrm{fd}})^{\times}$$

Remark 9.4. The topological cobordism hypothesis can be restated by saying that the (∞, d)-category $\mathscr{B}\mathrm{ord}^{\mathrm{fr}}_{(\infty,d)}$ is the *free symmetric monoidal (∞, d)-category with duals generated from a single object.*

9.2. **Geometric Field Theories.** The goal of this section is to both generalize and make precise what we did in the previous motivational chapter. In the following we shall simply write $\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)}$ instead of $\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d),\mathrm{glob}}$. In light of the definition of topological quantum field theories, the following seems most natural:

Definition 9.5. Let $\mathscr{C}$ be a smooth symmetric monoidal (∞, d)-category and let $\mathbf{S}$ be a d-dimensional geometric structure with isotopies. A *d-dimensional smooth $\mathscr{C}$-valued functorial field theory with geometry* $\mathbf{S}$ is a smooth symmetric monoidal ∞-functor $\mathfrak{F}\colon \mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)} \to \mathscr{C}$.

Example 9.6. It is folklore that the theory of Quantum mechanics may be encoded as a suitable one-dimensional quantum field theory where the corresponding bordism category is endowed with the geometric structure of Riemmanian metrics. In particular, special emphasis needs to be given to the target of our quantum field theory so as to properly encode quantum mechanics. The idea is the following. For a smooth functorial field theory $\mathfrak{F}\colon \mathfrak{Bord}^{\mathfrak{Riem}^{\mathfrak{f}}_1}_{(\infty,1)} \to \mathscr{C}$ where $\mathscr{C}$ is a suitable (smooth) symmetric monoidal $(\infty, 1)$-category (ideally with duals) of values. Typically we would assume $\mathscr{C}$ to be something like the $(\infty, 1)$-categorical version of the category of vector spaces (i.e. the Rezk nerve of Vect), or even Hilbert spaces etc. An object of $\mathfrak{Bord}^{\mathrm{Riem}^{\mathfrak{f}}_d}_{(\infty,1)}$, say a single point $\bullet$ is mapped to the state space $c := \mathscr{Z}(\bullet) \in \mathscr{C}$. A 1-morphism, that is, a bordism $[0, l]$ of length l is mapped to an automorphism $\mathfrak{F}_l\colon c \to c$. Functoriality then implies that if we have composable bordisms with lengths $[0, l], [l, l + l']$, then we have

$$\mathfrak{F}_{l+l'} \simeq \mathfrak{F}_{l'}\mathfrak{F}_l$$

which is precisely the time propagation property of the solution to the Schrödinger equation $l \mapsto e^{-i\hbar\mathfrak{H}l}$, where $\mathfrak{H}$ denotes the Hamiltonian of the given quantum mechanical system.

We recall that in our setting the notion of a U-family of d-framings was encoded by means of the representable enriched presheaf $よ(\mathbb{R}^d \times U \twoheadrightarrow U)$. We then note that $\mathfrak{Bord}^{よ(\mathbb{R}^d\times U\twoheadrightarrow U)}_{(\infty,d)}$ has a canonical object:

Definition 9.7. Let $d \geqslant 0$ and let $\{e_i\}_{i=1}^d$ be the standard orthonormal basis of $\mathbb{R}^d$. Let

$$\mathrm{pt} := \Big(\mathbb{R}^d,\ \{C^k \times U \mid 1 \leqslant k \leqslant d\},\ 1\colon \mathbb{R}^d \times U \to \langle 1\rangle,\ (\mathrm{id}_{\mathbb{R}^d\times U}, \mathrm{id}_U)\Big)$$

be the object in $\mathfrak{Bord}^{よ(\mathbb{R}^d\times U\twoheadrightarrow U)}_{(\infty,d)}(\mathbf{0}, \langle 1\rangle, U)$ with $C^k := \mathrm{span}\{e_i \mid i \neq k\}$, and $1\colon \mathbb{R}^d \times U \to \langle 1\rangle$ is the constant 1-function, while $(\mathrm{id}_{\mathbb{R}^d\times U}, \mathrm{id}_U)$ is the geometric structure given by the identity morphism in $\mathrm{FEmb}_d(\mathbb{R}^d \times U \twoheadrightarrow U,\ \mathbb{R}^d \times U \twoheadrightarrow U)$.

By the Yoneda Lemma the above object may be interpreted as a map

$$\mathrm{pt}\colon j(\mathbf{0},\langle 1\rangle,U)\to \mathfrak{Bord}_{(\infty,d)}^{\text{よ}(\mathbb{R}^d\times U\twoheadrightarrow U)}$$

This gives rise to an evaluation map

$$\begin{array}{ccc} \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,d)}^{\text{よ}(\mathbb{R}^d\times U\twoheadrightarrow U)},\mathscr{C})^{\times} & \overset{\mathfrak{eval}(U)}{\dashrightarrow} & \mathfrak{Map}(jU,\mathscr{C}^{\times}) \\ \Big\downarrow{\scriptstyle \mathrm{pt}^{\star}} & & \Big\uparrow{\scriptstyle \cong} \\ \mathrm{Fun}^{\otimes}(j(\mathbf{0},\langle 1\rangle,U),\mathscr{C})^{\times} & \xrightarrow[\cong]{} & \mathfrak{Map}(j(\langle 1\rangle,U),\mathscr{C}^{\times}) \end{array}$$

where the isomorphism $\mathrm{Fun}^{\otimes}(j(\mathbf{0},\langle 1\rangle,U),\mathscr{C})^{\times}\to\mathfrak{Map}(j(\langle 1\rangle,U),\mathscr{C}^{\times})$ is induced by the adjunction from Lemma 7.134 for $\mathbf{m}=\mathbf{0}$, while the other isomorphism follows from the fact that $j\langle 1\rangle$ is the monoidal unit with respect to the Day convolution tensor product. Evaluation at pt leads to the the *Geometric Framed Cobordism Hypothesis*:

Theorem 9.8 (Geometric Framed Cobordism Hypothesis [17]). *Fix $d\geqslant 0, U\in\mathrm{Cart}$, and let $\mathscr{C}$ be a smooth symmetric monoidal (∞,d)-category with duals. The smooth symmetric monoidal (∞,d)-category $\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,d)}^{\text{よ}(\mathbb{R}^d\times U\twoheadrightarrow U)},\mathscr{C})$ is a smooth symmetric monoidal ∞-groupoid, i.e., the inclusion of the core yields a weak equivalence in $\mathscr{C}^{\infty}\mathrm{Cat}_{(\infty,d)}^{\otimes,\dagger}$:*

$$\mathfrak{L}_{\{1,\ldots,d\}}\Big(\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,d)}^{\text{よ}(\mathbb{R}^d\times U\twoheadrightarrow U)},\mathscr{C})^{\times}\Big)\xrightarrow{\simeq}\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,d)}^{\text{よ}(\mathbb{R}^d\times U\twoheadrightarrow U)},\mathscr{C})$$

where $\mathfrak{L}_{\{1,\ldots,d\}}$ was defined right before Lemma 7.135. Furthermore, evaluation at the point (see 9.7) yields an equivalence of smooth symmetric monoidal ∞-groupoids

$$\mathfrak{eval}(U)\colon \mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,d)}^{\text{よ}(\mathbb{R}^d\times U\twoheadrightarrow U)},\mathscr{C})^{\times}\xrightarrow{\simeq}\mathfrak{Map}(jU,\mathscr{C}^{\times})$$

Remark 9.9. We note that if we evaluate the above weak equivalence at $U=\mathbb{R}^0$, we obtain an equivalence

$$\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,d)}^{\text{よ}(\mathbb{R}^d\times U\twoheadrightarrow U)},\mathscr{C})^{\times}(\mathbb{R}^0)\xrightarrow{\simeq}\mathfrak{Map}(jU,\mathscr{C}^{\times})(\mathbb{R}^0)\cong\mathscr{C}^{\times}(U)$$

which gives a U-family version of Lurie's topological cobordism hypothesis (Theorem 9.2).

Theorem 9.10 (Geometric Cobordism Hypothesis [17]). *Let $d\geqslant 0$, and fix a smooth symmetric monoidal (∞,d)-category $\mathscr{C}$ with duals and a d-dimensional geometric structure $\mathbf{S}$ with isotopies. Let $\mathscr{C}_d^{\times}:\mathrm{FEmb}_d^{\mathrm{op}}\to\mathrm{Psh}_{\Delta}(\Gamma\times\mathrm{Cart})$ be the (fibrant) simplicial presheaf with values in smooth symmetric monoidal ∞-groupoids defined by*

$$(M\twoheadrightarrow U)\mapsto\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,d)}^{\text{よ}(M\twoheadrightarrow U)},\mathscr{C})^{\times}$$

The smooth symmetric monoidal (∞,d)-category $\mathrm{Fun}^{\otimes}(\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)},\mathscr{C})$ is a smooth symmetric monoidal ∞-groupoid, i.e., the inclusion of the core yields a weak equivalence in $\mathscr{C}^{\infty}\mathrm{Cat}_{(\infty,d)}^{\otimes,\dagger}$:

$$c_{\{1,\ldots,d\}}\Big(\mathrm{Fun}^{\otimes}(\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)},\mathscr{C})^{\times}\Big)\xrightarrow{\simeq}\mathrm{Fun}^{\otimes}(\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)},\mathscr{C})$$

Furthermore, we have a natural weak equivalence (in fact, an isomorphism)

$$\mathrm{Fun}^{\otimes}(\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)},\mathscr{C})^{\times} \simeq \mathfrak{Map}_{\mathfrak{FEmb}_d}(\mathbf{S},\mathscr{C}^{\times}_d)$$

Proof. By the enriched version of Corollary 6.43 and Proposition 8.22 we know that any geometric structure $\mathbf{S}$ may be written as a homotopy colimit

$$\mathbf{S} \simeq \underset{\mathbb{R}^d\times U\to\mathbf{S}}{\mathrm{hocolim}}\ \text{よ}(\mathbb{R}^d\times U \twoheadrightarrow U)$$

We then calculate

$$\begin{aligned}
\mathrm{Fun}^{\otimes}(\mathfrak{Bord}^{\mathbf{S}}_{(\infty,d)},\mathscr{C}) &\simeq \mathrm{Fun}^{\otimes}(\mathfrak{Bord}^{\mathrm{hocolim}_{\mathbb{R}^d\times U\to\mathbf{S}}\ \text{よ}(\mathbb{R}^d\times U\twoheadrightarrow U)}_{(\infty,d)},\mathscr{C})\\
&\simeq \underset{\mathbb{R}^d\times U\to\mathbf{S}}{\mathrm{holim}}\ \mathrm{Fun}^{\otimes}(\mathfrak{Bord}^{\text{よ}(\mathbb{R}^d\times U\twoheadrightarrow U)}_{(\infty,d)},\mathscr{C})\\
&\simeq \underset{\mathbb{R}^d\times U\to\mathbf{S}}{\mathrm{holim}}\ \mathscr{C}^{\times}_d(\mathbb{R}^d\times U\twoheadrightarrow U)\\
&\simeq \underset{\mathbb{R}^d\times U\to\mathbf{S}}{\mathrm{holim}}\ \mathfrak{Map}(\text{よ}(\mathbb{R}^d\times U\twoheadrightarrow U),\mathscr{C}^{\times}_d)\\
&\simeq \underset{\mathbb{R}^d\times U\to\mathbf{S}}{\mathrm{holim}}\ \int_{N\twoheadrightarrow V}\mathfrak{Map}(\text{よ}(\mathbb{R}^d\times U\twoheadrightarrow U)(N\twoheadrightarrow V),\mathscr{C}^{\times}_d(N\twoheadrightarrow V))\\
&\simeq \underset{\mathbb{R}^d\times U\to\mathbf{S}}{\mathrm{holim}}\ \mathfrak{Map}_{\mathfrak{FEmb}_d}(\mathbb{R}^d\times U\twoheadrightarrow U,\mathscr{C}^{\times}_d)\\
&\simeq \mathfrak{Map}_{\mathfrak{FEmb}_d}(\mathbf{S},\mathscr{C}^{\times}_d)
\end{aligned}$$

□

The geometric cobordism hypothesis is therefore really all about *smooth spaces* of field theories and it tells us that we can calculate these by instead calculating the simpler objects $\mathfrak{Map}_{\mathfrak{FEmb}_d}(\mathbf{S},\mathscr{C}^{\times}_d)$. In practice we can use the following scheme to calculate $\mathfrak{Map}_{\mathfrak{FEmb}_d}(\mathbf{S},\mathscr{C})$:

- (i) Guess a candidate $\mathscr{D}\colon \mathfrak{FEmb}^{\mathrm{op}}_d \to \mathrm{Psh}_{\Delta}(\Gamma\times\mathrm{Cart})$ which satisfies the descent condition with respect to $\mathfrak{FEmb}_d$.
- (ii) Write down any natural map $\mathfrak{W}\colon \mathscr{D}\to\mathscr{C}^{\times}_d$.
- (iii) Prove that the composition of maps

$$\begin{array}{ccc}
\mathscr{D}(\mathbb{R}^d\times U\twoheadrightarrow U) & \xrightarrow{\ \mathfrak{W}\ } & \mathscr{C}^{\times}_d(\mathbb{R}^d\times U\twoheadrightarrow U)\simeq \mathrm{Fun}^{\otimes}(\mathfrak{Bord}^{\text{よ}(\mathbb{R}^d\times U\twoheadrightarrow U)}_{(\infty,d)},\mathscr{C})\\
 & \searrow & \downarrow \mathfrak{eval}(U)\\
 & & \mathfrak{Map}(jU,\mathscr{C}^{\times})
\end{array}$$

  is a weak equivalence.
- (iv) From this we deduce (by the 2-out-of-3 property) that $\mathfrak{W}$ is a local weak equivalence.

Example 9.11. Consider the 0-dimensional bordism category $\mathfrak{Bord}^{\mathscr{C}^{\infty}(-,\mathfrak{X})}_{(\infty,0)}$ endowed with the (unenriched) geometric structure from Example 8.9. Moreover, denote by $\mathfrak{V}$ the smooth symmetric monoidal ∞-groupoid given by the assignment

$$(\langle l\rangle,U)\mapsto \mathfrak{V}(\langle l\rangle,U):=\mathscr{C}^{\infty}(U,\mathbb{R})^l$$

where the set $\mathscr{C}^{\infty}(U,\mathbb{R})$ is interpreted as a constant simplicial set. Noting that $\mathfrak{FEmb}\mathrm{Cart}_0=\mathrm{Cart}$, the geometric cobordism hypothesis states that

$$\mathrm{Fun}^{\otimes}(\mathfrak{Bord}^{\mathscr{C}^{\infty}(-,\mathfrak{X})}_{(\infty,0)},\mathfrak{V})\simeq \mathfrak{Map}_{\mathrm{Cart}}(\mathscr{C}^{\infty}(-,\mathfrak{X}),\mathfrak{V})$$

$$\simeq \int_{U \in \mathrm{Cart}} \mathfrak{Map}(\mathscr{C}^\infty(U, \mathfrak{X}), \mathfrak{V}(U))$$

The Yoneda lemma then implies

$$\mathrm{Fun}^{\otimes}(\mathfrak{Bord}_{(\infty,0)}^{\mathscr{C}^\infty(-,\mathfrak{X})}, \mathfrak{V}) \simeq \mathfrak{V}(\mathfrak{X}) = \mathscr{C}^\infty(\mathfrak{X}, \mathbb{R})^\bullet$$

Hence, the smooth space of 0-dimensional smooth functorial field theories with geometry $\mathscr{C}^\infty(-, \mathfrak{X})$ is the space of smooth functions from $\mathfrak{X}$ to $\mathbb{R}$.

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